Progress in Nonlinear Differential Equations
and Their Applications
82

Antonio Ambrosetti
David Arcoya

# An Introduction to Nonlinear Functional Analysis and Elliptic Problems

Birkhäuser

Birkhäuser

# Progress in Nonlinear Differential Equations and Their Applications

Volume 82



For other titles published in this series, go to
http://www.springer.com/series/4889

Antonio Ambrosetti • David Arcoya

# An Introduction to Nonlinear Functional Analysis and Elliptic Problems

Birkhäuser
Boston • Basel • Berlin

Antonio Ambrosetti
SISSA-International School
for Advanced Studies
34136 Trieste
Italy
ambr@sissa.it

David Arcoya
Departamento de Análisis Matemático
Universidad de Granada
18071 Granada
Spain
darcoya@ugr.es

ISBN 978-0-8176-8113-5 e-ISBN 978-0-8176-8114-2
DOI 10.1007/978-0-8176-8114-2
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011930398

Mathematics Subject Classification (2011): 35-01, 46-01, 49-01



Printed on acid-free paper

Birkhäuser is part of Springer Science+Business Media (www.springer.com)

# Preface

The purpose of this book is to introduce the reader to some of the main abstract tools of nonlinear functional analysis and their applications to semilinear elliptic Dirichlet boundary value problems.

In the first chapter we outline some general results on Fréchet differentiability, Nemitski operators, weak and strong solutions of the linear Laplace equation, linear compact operators and their eigenvalues and Sobolev spaces. This last topic is discussed in greater generality in Appendix A.

Chapter 2 deals with the Banach contraction principle and with a fixed point theorem for increasing operators. In Chap. 3 we study the local inversion theorem, the Hadamard–Caccioppoli global inversion theorem and the case in which the map to be inverted has fold singularities. Chapter 4 is concerned with the Leray–Schauder topological degree. Variational methods are discussed in Chap. 5. Minima, the mountain pass theorem and the linking theorem are stated and proved. Chapter 6 deals with local and global bifurcation theory.

The abstract results collected in first part of the book are applied in the second part to prove existence and multiplicity results for semilinear elliptic Dirichlet boundary value problems on bounded domains in $\mathbb{R}^N$. We emphasize that the choice of the appropriate abstract tool depends on the behavior of the nonlinearity $f$ as well as on the kind of results one expects.

First, in Chap. 7, we outline how a semilinear elliptic boundary value problem can be transformed into an operator equation in an appropriate Banach or Hilbert function space. In Chap. 8 we consider the case in which, roughly, $f$ is sublinear at infinity and one can prove a priori estimates for possible solutions. In this case one can use degree theory or variational methods or the global inversion theorem. Chapter 9 deals with asymptotically linear problems, for which one can also use several different approaches such as global bifurcation or variational methods. In Chap. 10 we study problems with asymmetric nonlinearities, when the behaviors at $+\infty$ and $-\infty$ are different. If one aims to find the precise number of multiple solutions, the most appropriate approach turns out to be the global inversion theorem in the presence of fold singularities. But one can also use sub- and super-solutions jointly with degree theoretical arguments. Nonlinearities that are superlinear at infinity are considered in Chap. 11 by means of the mountain pass or linking theorems.

In all of the preceding chapters we do not consider more general sophisticated versions of problems, but prefer to study model cases containing the main features of the arguments without unnecessary technical details.

The last two chapters of the book are concerned with slightly more advanced topics of current research. In Chap. 12 a class of quasilinear elliptic problems is discussed using critical point theory. Here the corresponding Euler functional is not $C^1$, and hence a new form of the mountain pass theorem has to be proved. Chapter 13 deals with nonlinear Schrödinger equations on $R^N$. We prove the existence of ground and bound states as well as semiclassical states.

The book is addressed to senior undergraduate and graduate students of mathematics as well as to students of applied sciences, who wish to utilize a modern approach to the fascinating topic of nonlinear elliptic partial differential equations.

# Contents

# Notation

- For every $s \in \mathbb{R}$ we consider the positive and negative parts given by $s^+ = \max\{s, 0\}$ and $s^- = \min\{s, 0\}$. $C, C_1, C_2, \ldots$ denote possibly different positive constants.
- If $\Omega$ is a measurable set in $\mathbb{R}^N$, we denote by $|\Omega|$ the Lebesgue measure of $\Omega$ and by $\int u$ the integral of a function $u$ in $\Omega$. Hence, unless it is explicitly stated, the integrals are always understood to be on $\Omega$.
- $\omega \subset\subset \Omega$ denotes that $\omega$ is compactly embedded in $\Omega$, that is, the closure $\overline{\omega}$ of $\omega$ is a compact subset of $\Omega$.
- If $\Omega$ is an open set in $\mathbb{R}^N$ and $\alpha$ is a multi-index, namely $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, with $\alpha_i$ a non-negative integer, we denote by $D^\alpha u$ the partial derivative $\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}$, where $|\alpha| = \sum_{i=1}^{N} \alpha_i$ is the order of $\alpha$. The set of all infinitely-differentiable functions of compact support in $\Omega$ is represented by $C_0^\infty(\Omega)$.
- For a non-negative integer $k$ and $0 < \alpha \leq 1$, we denote by $C^{k,\alpha}(\overline{\Omega})$ the space of the functions whose derivatives up to order $k$ are $\alpha$-Hölder continuous in $\overline{\Omega}$. In particular, we write $C^k(\overline{\Omega})$ if $\alpha = 0$. Moreover, $C_0^1(\overline{\Omega})$ is the space of all functions of class $C^1$ in an open neighborhood of $\Omega$ such that they vanish at the boundary $\partial\Omega$ of $\Omega$.
- For $1 \leq p \leq +\infty$, $\|u\|_p$ is the usual norm of a function $u \in L^p(\Omega)$.
- We have equipped the standard Sobolev space $H_0^1(\Omega)$ with the norm $\|u\| = \left(\int |\nabla u|^2\right)^{1/2}$.
- We also denote by

$$2^* = \begin{cases} \frac{2N}{N-2}, & \text{if } N \geq 3 \\ +\infty, & \text{if } N \leq 2 \end{cases}$$

  the critical Sobolev exponent and by $\mathcal{S} = \sup\{\|u\|_{2^*} : \|u\| = 1\}$ the Sobolev embedding constant.
- $\overline{2} = 2N/(N+2)$ is the Hölder conjugate exponent of $2^*$.
- The truncature functions $T_k$ and $G_k$ are given by

$$T_k(s) = \max\{\min\{s, k\}, -k\} \text{ and } G_k(s) = s - T_k(s),$$

  for every $s \in \mathbb{R}$.

- We denote a Banach (resp. Hilbert) space with the letter $X$ (resp. $E$). The identity operator is denoted by $I$. The functionals, i.e., (nonlinear) operators from a Banach space $X$ to $\mathbb{R}$, are denoted by letters $\mathcal{J}, \mathcal{H}, \mathcal{I}, \ldots$ In general, the operators between different Banach spaces $X$ and $Y$ are denoted by letters $F, G, \ldots$, while letters $T, S, \ldots$ are used for operators from a Banach space into itself.
- The weak convergence of a sequence $w_n$ in a Banach space to $w$ will be denoted $w_n \rightharpoonup w$.
- If $F : X \longrightarrow Y$ is an operator between Banach spaces, we denote $\operatorname{Ker} F = \{u \in X : F(u) = 0\}$ and $\operatorname{Range} F = \{F(x) : x \in X\}$.
- $0 < \lambda_1 < \lambda_2 \leq \lambda_3, \ldots$ denote the eigenvalues of $-\Delta u = \lambda u$, $u \in E$ and $\varphi_i$ satisfies $-\Delta\varphi_i = \lambda_i\varphi_i$ with $\|\varphi_i\| = 1$ and $(\varphi_i \mid \varphi_j) = 0$ for $i \neq j$. We take $\varphi_1 > 0$.

# Chapter 1
# Preliminaries

In this chapter we collect some preliminary results that we will use throughout the rest of the book, such as Fréchet derivatives, superposition operators and weak and classical solutions of linear elliptic equations and their eigenvalues. Sobolev function spaces are also outlined, although a more complete treatment is postponed until Appendix A.

## 1.1 Sobolev Spaces

In the classical study of boundary value problems associated to a differential equation it is usual to add to the "local" space (in $\Omega$) in which we are searching its solution, for instance, $C^k(\Omega)$, some "global" condition. For instance, it may be required that $u \in C(\overline{\Omega})$ or, in some cases, $u \in C^k(\overline{\Omega})$. Similarly to the construction of $C^k(\overline{\Omega})$ from the local space $C^k(\Omega)$ by imposing the global condition (in $\Omega$) of continuity in $\overline{\Omega}$ of the function and its derivatives up to the order $k$, the construction of the Sobolev spaces $W^{k,p}(\Omega)$ is a combination of local properties (weak derivatives in $L^1_{\text{loc}}(\Omega)$) together with a suitable "global" condition in $\Omega$ (the weak derivatives belong to $L^p(\Omega)$).

**Definition 1.1.1** If $\Omega \subset \mathbb{R}^N$ is an open subset, $p \in [1, +\infty]$ and $k \in \mathbb{N}$, then the Sobolev space $W^{k,p}(\Omega)$ is defined as the space of the functions $u \in L^p(\Omega)$ such that for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ with order $|\alpha| \le k$ there exists a function $v_\alpha \in L^p(\Omega)$ satisfying

$$\int \varphi(x) v_\alpha(x) dx \;=\; (-1)^{|\alpha|} \int u(x) D^\alpha \varphi(x) dx, \;\; \forall \varphi \in C_0^\infty(\Omega).$$

Here and in the sequel, unless it is explicitly stated, the integrals are always understood to be on $\Omega$. The function $v_\alpha$ is called the weak derivative of $u$ of order $\alpha$ and

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82,
DOI 10.1007/978-0-8176-8114-2_1, © Springer Science+Business Media, LLC 2011

is denoted by $D^\alpha u$. $W^{k,p}(\Omega)$ can be equipped with two equivalent norms:

$$\|u\|_{k,p} \equiv \begin{cases} \left[\sum_{|\alpha|\le k} \|D^\alpha u\|_p^p\right]^{1/p} & \text{if } p \in [1,+\infty) \\ \max_{|\alpha|\le k} \|D^\alpha u\|_\infty & \text{if } p = +\infty. \end{cases}$$

$$|||u|||_{k,p} \equiv \sum_{|\alpha|\le k} \|D^\alpha u\|_p.$$

A particular case is the space $\left(H^k(\Omega) \equiv W^{k,2}(\Omega), \|\cdot\|_{k,2}\right)$ because the norm $\|\cdot\|_{k,2}$ is the one associated to the inner product

$$(u,v)_{k,2} = \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}\,.$$

Since the product space of copies of $L^p(\Omega)$ is complete, one deduces (see Exercise 1) that $W^{k,p}(\Omega)$ is a Banach space for $1 \le p \le +\infty$, it is reflexive for $1 < p < +\infty$ and it is separable for $1 \le p < +\infty$. In particular, $H^k(\Omega)$ is a separable Hilbert space.

It is clear that every function in $C^k(\Omega)$, such that its partial derivatives up to order $k$ are also in $L^p(\Omega)$, belongs to $W^{k,p}(\Omega)$. Meyers and Serrin proved that this class of functions is a dense subspace of $W^{k,p}(\Omega)$ provided that $1 \le p < +\infty$ (see Theorem A.2.5).

In the study of the classical Dirichlet problem associated to the Laplace equation in an open $\Omega \subset \mathbb{R}^N$, the value of the solution on $\partial\Omega$ is prescribed. Thus, to give a weak formulation of the Dirichlet problem, we need to define the value of $u \in W^{k,p}(\Omega)$ on $\partial\Omega$. This is not trivial at all because $u \in L^p(\Omega)$ is the equivalence class of the functions which are equal almost everywhere in $\Omega$. We first discuss the simplest case: What is the weak space similar to the space of $C^k(\overline{\Omega})$-functions satisfying $u = 0$ on $\partial\Omega$? To answer this question we introduce a new space.

**Definition 1.1.2** If $\Omega \subset \mathbb{R}^N$ is open, $1 \le p \le +\infty$ and $k \in \mathbb{N}$, we denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the space $W^{k,p}(\Omega)$. In the particular case $p = 2$, we also write $W_0^{k,p}(\Omega) = H_0^k(\Omega)$.

Observe that $W_0^{k,p}(\Omega)$ with the induced norm of $W^{k,p}(\Omega)$ is a Banach space for $1 \le p \le +\infty$, it is reflexive provided that $1 < p < +\infty$ and it is separable if $1 \le p < +\infty$. In particular, $H_0^k(\Omega)$ is a separable Hilbert space.

In general, the strict inclusion $W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ holds. Roughly, the smaller $\mathbb{R}^N \setminus \Omega$ is, the smaller $W^{k,p}(\Omega) \setminus W_0^{k,p}(\Omega)$ is (see [1, Theorem 3.31]). In particular, if $\Omega = \mathbb{R}^N$ then $W_0^{k,p}(\mathbb{R}^N) = W^{k,p}(\mathbb{R}^N)$ (see Proposition A.3.10).

One of the main properties of $W_0^{k,p}(\Omega)$ is the well-known Poincaré inequality: *If $p \in [1,+\infty)$ and $\Omega \subset \mathbb{R}^N$ is open and bounded in one direction, then there exists a positive constant $C$ depending uniquely on $\Omega$ such that*

$$C\int |u|^p \le \int |\nabla u|^p, \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.1}$$

As a consequence, under the hypotheses of the Poincaré inequality, $\|\nabla u\|_p$ defines a norm $W_0^{1,p}(\Omega)$ which is equivalent to $\|\cdot\|_{1,p}$. In addition, if $p = 2$, this new norm $\|\nabla u\|_2$ in $H_0^1(\Omega)$ is associated to the inner product $\int \nabla u \cdot \nabla v$ for $u, v \in H^1(\Omega)$.

We introduced the space $W_0^{k,p}(\Omega)$ to obtain the weak formulation of the functions that vanish on the boundary $\partial\Omega$. Using this space, it is easy to state what we understand by an ordering between values on the boundary of functions in $W^{k,p}(\Omega)$. Specifically, if $1 \le p < +\infty$, $k \in \mathbb{R}$ and $u, v \in W^{1,p}(\Omega)$, we say that

- $u \le k$ on $\partial\Omega \iff (u-k)^+ = \max\{u-k, 0\} \in W_0^{1,p}(\Omega)$.
- $u \ge k$ on $\partial\Omega \iff -u \le -k$ on $\partial\Omega$.
- $u \le v$ on $\partial\Omega \iff u - v \le 0$ on $\partial\Omega$.
- $u \ge v$ on $\partial\Omega \iff v \le u$ on $\partial\Omega$.
- $u = v$ on $\partial\Omega \iff \begin{cases} u \le v \text{ on } \partial\Omega \\ v \le u \text{ on } \partial\Omega. \end{cases}$

## *1.1.1 Embedding Theorems*

We state below the well-known Sobolev and Rellich–Kondrachov embedding theorems. We say that the normed space $(X, \|\cdot\|_X)$ is embedded in the normed space $(Y, \|\cdot\|_Y)$, and we denote it by $X \hookrightarrow Y$, if there exists an injective linear and continuous operator $I$ from $X$ into $Y$. In this case, the operator is called an embedding. We say that the space $X$ is compactly embedded in the space $Y$, if there exists an embedding of $X$ in $Y$ which is compact. An operator $T : X \to X$ is said to be compact if it is continuous and $T(A)$ is relatively compact for all bounded sets $A \subset X$ (see Definition 1.3.1 below). We state in a unique theorem a unified version of the Sobolev and Rellich–Kondrachov theorems (see Theorems A.4.3 and A.4.9 for more general results).

**Theorem 1.1.3** *If $\Omega \subseteq \mathbb{R}^N$ is an open subset with boundary $\partial\Omega$ of class $C^1$, $k \in \mathbb{N}$ and $1 \le p < \infty$, then*

1. *If $k < \frac{N}{p}$ then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \in \left[p, \frac{Np}{N-kp}\right]$.*
2. *If $k = \frac{N}{p}$ then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $p \le q < \infty$.*
3. *If $k > \frac{N}{p}$ then $W^{k,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$, where*

$$\alpha = \begin{cases} k - \frac{N}{p}, & \text{if } k - \frac{N}{p} < 1, \\ \text{every } \alpha \in [0,1), & \text{if } k - \frac{N}{p} = 1, \\ 1, & \text{if } k - \frac{N}{p} > 1. \end{cases}$$

*If in addition $\Omega$ is bounded, all the above embeddings are compact except for $q = \frac{Np}{N-kp}$ in case 1.*

*Furthermore, if we replace the space $W^{k,p}(\Omega)$ by $W_0^{k,p}(\Omega)$, all the embeddings (also the compact ones) hold without necessity of assuming the regularity of the boundary $\partial\Omega$ of $\Omega$.* □

## 1.2 Linear Elliptic Equations

Many applications lead to the study of minimization problems like

$$\min \int H(x, v, \nabla v)\, dx$$

where $\Omega$ is an open subset of $\mathbb{R}^N$ and $H$ is a function in $\Omega \times \mathbb{R}^{N+1}$.

It is more convenient to solve this problem in the weak formulation than in the classical one. In order to do this in a more clear way, we consider the Dirichlet principle, i.e., the problem which consists in looking for the function $u$ such that $\nabla u$ has minimal $L^2$-norm in the manifold of all functions with prescribed value $u_0$ on $\partial\Omega$:

$$\min \left\{ \int |\nabla v|^2\, dx \ : \ v = u_0 \text{ on } \partial\Omega \right\}. \tag{1.2}$$

Now, thinking that the simpler similar minimization problem

$$\min \left\{ |v|^2 \ : \ v \in \mathbb{Q},\ v^2 \geq 2 \right\}$$

has no solution because $\mathbb{Q}$ is not complete or equivalently because the set $\left\{ v \in \mathbb{Q} \ : \ v^2 \geq 2 \right\}$ is not closed in the Banach space $\mathbb{R}$, we understand that it is a good idea to set out minimization in the completion (with respect to the $L^2(\Omega)$-norm of the gradient) of the functions with value $u_0$ on $\partial\Omega$. This means that we study the Dirichlet principle as

$$\min_{v \in A} \mathcal{J}(v)$$

with the functional $\mathcal{J} : X = H^1(\Omega) \longrightarrow \mathbb{R}$ given by $\mathcal{J}(v) = \int |\nabla v|^2\, dx$ and $A = \{v \in H^1(\Omega) \ : \ v - u_0 \in H_0^1(\Omega)\}$, for some $u_0 \in H^1(\Omega)$.

### *1.2.1 Fréchet Differentiability*

Let $X, Y$ be Banach spaces, $u \in X$ and consider a map $F : X \mapsto Y$. In the particular case that $Y = \mathbb{R}$, $F$ is called a *functional*. We say that $F$ is differentiable at $u \in X$ along the direction $v \in X$ if there exists

$$L_u[v] := \lim_{t \to 0} \frac{F(u + tv) - F(u)}{t}.$$

Elementary examples on $X = \mathbb{R}^2$ show that $F$ can be differentiable along every direction without being continuous.

We say that $F$ is (Fréchet) differentiable at $u \in X$ if there exists a linear continuous map $L_u : X \longrightarrow Y$ such that

$$F(u+v) - F(u) = L_u[v] + o(\|v\|), \quad \text{as } \|v\| \to 0.$$

The map $L_u$ is uniquely determined by $F$ and $u$ and will be denoted by $dF(u)$ or else $F'(u)$. It is easy to see that if $F$ is Fréchet differentiable, then it is also differentiable along any direction. Conversely, if $F$ is differentiable along any directions, $L_u \in L(X,Y)$ and the map $u \mapsto L_u$ is continuous from $X$ to $L(X,Y)$, then $F$ is Fréchet differentiable.

The Fréchet derivative has the same properties as the usual differential in Euclidean spaces. For example, if $X, Y, Z$ are Banach spaces, $F : X \mapsto Y$, $G : Y \mapsto Z$ and $F$ is differentiable at $u \in X$, resp. $G$ is differentiable at $F(u) \in Y$, then the composite map $G \circ F$ is differentiable at $u$ and the following chain rule holds:

$$D(G \circ F)(u)[v] = dG(F(u))[dF(u)[v]].$$

One can also define higher derivatives, partial derivatives and so on. In particular, the second derivative will be denoted by $d^2F(u)$. For more details, the reader is referred, e.g., to Chaps. 1 and 2 of [17].

### *1.2.2 Nemitski Operators*

Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$. The Nemitski operator associated to $f(x,u)$ is the superposition operator

$$f : u(x) \mapsto f(x, u(x))$$

defined on the class of measurable functions $u : \Omega \longrightarrow \mathbb{R}$. If there is no possible misunderstanding, we will use the same symbol $f$ to denote the Nemitski operator associated to $f(x,u)$. Here and below $\Omega$ denotes a bounded domain in $\mathbb{R}^N$.

Suppose that $f(x,u)$ is Carathéodory, namely:

1. $f(x,.)$ is continuous in $\mathbb{R}$ for a.e. $x \in \Omega$,
2. $f(.,u)$ is measurable in $\Omega$ for all $u \in \mathbb{R}$.

Let us point out that if $f(x,u)$ is Carathéodory then the Nemitski operator $f$ maps any measurable function $u(x)$ to a measurable function $f(u)$. The continuity and Fréchet differentiability of Nemitski operators are collected in the following theorems. We omit the proof, referring the reader to Sect. 1.2 of [17].

**Theorem 1.2.1** *Suppose that $f(x,u)$ is Carathéodory and that there exist $a, b \in \mathbb{R}$ such that*

$$|f(x,u)| \le a + b|u|^{p/q}, \quad p, q \ge 1.$$

*Then the Nemitski operator $f$ is continuous from $L^p(\Omega)$ to $L^q(\Omega)$.* □

**Theorem 1.2.2** *Suppose that $f(x,u)$ is Carathéodory and that $f(x,.)$ is differentiable with respect to $u$ with derivative $f_u(x,u)$ which is Carathéodory. Moreover, let $p > 2$ and suppose that there exist $c, d \in \mathbb{R}$ such that*

$$|f_u(x,u)| \leq c + d|u|^{p-2}.$$

*Then the Nemitski operator $f$ is differentiable on $L^p(\Omega)$ with differential $df(u) : v \mapsto f_u(u)v$.* □

*Remark 1.2.3* (i) It is possible to prove that if the Nemitski operator $f$ maps $L^p(\Omega)$ into $L^q(\Omega)$ then $f \in C(L^p(\Omega), L^q(\Omega))$.

(ii) Let $f(x,u)$ and $f_u(x,u)$ be Carathéodory functions and $|f_u(x,u)| \leq$ const. Then one can show that $f : L^2(\Omega) \longrightarrow L^2(\Omega)$ is differentiable along every direction. On the other hand, if $f$ is Fréchet differentiable at some $u^* \in L^2(\Omega)$ then there exist measurable functions $a(x), b(x)$ such that $f(x,u) = a(x) + b(x)u$.

### *1.2.3 Dirichlet Principle*

Problem (1.2) is solved by using the Weierstrass theorem.

**Theorem 1.2.4** *Let $\mathcal{J} : A \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous functional defined on a compact topological space $A$. Then $\mathcal{J}$ is bounded from below and it attains its minimum.*

*Proof* We only give here the proof for the case in which $\mathcal{J}$ is sequentially lower semicontinuous[1] and $A$ is sequentially compact. Define

$$\alpha = \begin{cases} \inf\{\mathcal{J}(v) : v \in A\}, & \text{if this infimum exists} \\ -\infty, & \text{if the above infimum does not exist.} \end{cases}$$

In any case, we can choose a sequence $\{v_n\}$ in $A$ such that

$$\lim_{n\to+\infty} \mathcal{J}(v_n) = \alpha.$$

Since $A$ is sequentially compact, there exists a converging subsequence, still denoted $\{v_n\}$, to some point $v \in A$. Thus the sequentially lower semicontinuity of $\mathcal{J}$ implies that

$$\mathcal{J}(v) \leq \liminf_{n\to+\infty} \mathcal{J}(v_n) = \lim_{n\to+\infty} \mathcal{J}(v_n) = \alpha.$$

By the definition of $\alpha$ we get

$$\mathcal{J}(v) = \alpha,$$

which, in particular, means that $\alpha$ is finite, i.e., $\mathcal{J}$ is bounded from below. In addition, the infimum $\alpha$ of $\mathcal{J}$ is attained at $v$. □

[1] Let us remark that every sequentially lower semicontinuous functional $\mathcal{J}$ is also lower semicontinuous and that, in addition, the converse holds provided that $A$ satisfies the first axiom of countability. See [48] for the proofs of these facts and also for a complete proof of the theorem of Weierstrass.

If the dimension of $X$ is infinite, the hypothesis on the compactness of $A$ is very restrictive provided that we consider the topology induced by the norm. To overcome this difficulty we assume that $X$ is reflexive and $\mathcal{J}$ is coercive, i.e. $\lim_{u\in A,\, \|u\|\to+\infty} \mathcal{J}(u) = +\infty$. Indeed, the coerciveness allows us to reduce the minimization in $A$ to $A \cap \overline{B}(0, R)$ for some $R > 0$ large enough. Since the closed ball $\overline{B}(0, R)$ in a reflexive space is weakly compact, it is easy to deduce the following result.

**Corollary 1.2.5** *If $X$ is a reflexive Banach space, $A$ is a weakly closed subset in $X$ and $\mathcal{J} : A \longrightarrow \mathbb{R}$ is a w.l.s.c.*[2] *coercive functional in $A$, then there exists $u \in A$ such that*

$$\mathcal{J}(u) = \min\{\mathcal{J}(v) \ : \ v \in A\}. \qquad \square$$

Now, we are ready to prove the Dirichlet principle.

**Corollary 1.2.6** (Dirichlet principle) *Let $\Omega$ be an open bounded set in $\mathbb{R}^N$. For every fixed $u_0 \in H^1(\Omega)$, there exists a unique function $u \in H^1(\Omega)$ satisfying that $u - u_0 \in H_0^1(\Omega)$ and*

*(i) If $A = \{v \in H^1(\Omega) \ : \ v - u_0 \in H_0^1(\Omega)\}$, then*

$$\int |\nabla u|^2 = \min_{v\in A} \int |\nabla v|^2$$

*(ii) $u$ satisfies*

$$\int \nabla u \cdot \nabla v = 0, \quad \forall v \in H_0^1(\Omega),$$
$$u = u_0 \text{ on } \partial\Omega.$$

*Remark 1.2.7* A function $u \in H^1(\Omega)$ satisfying (ii) of the above corollary is called a weak solution for the boundary value problem (in the sequel, b.v.p.)

$$\begin{aligned} -\Delta u &= 0, \quad x \in \Omega \\ u &= u_0, \quad x \in \partial\Omega. \end{aligned}$$

In general, we have the following definition.

**Definition 1.2.8** Given $h \in L^2(\Omega)$, we say that $u \in H^1(\Omega)$ is a weak solution of the problem

$$\begin{aligned} -\Delta u &= h, \quad x \in \Omega \\ u &= u_0, \quad x \in \partial\Omega \end{aligned}$$

if it satisfies

$$\int \nabla u \cdot \nabla v = \int h\, v, \quad \forall v \in H_0^1(\Omega),$$

and $u = u_0$ on $\partial\Omega$.

---

[2] Although the notion of semicontinuity had been previously used in other fields, e.g., the Lebesgue integral, it was L. Tonelli who introduced this notion for the first time in the calculus of variations.

If $N \geq 3$, by the Sobolev embedding (see Theorem 1.1.3), every function $h \in L^{2N/(N+2)}(\Omega)$ belongs to the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$. Then one can substitute the space $L^2(\Omega)$ by $L^{2N/(N+2)}(\Omega)$ in the previous definition.

*Proof of Corollary 1.2.6* Consider $X := H^1(\Omega)$ and define the Dirichlet functional $\mathcal{J} : X \longrightarrow \mathbb{R}$ by taking

$$\mathcal{J}(v) = \int |\nabla v|^2, \quad v \in X.$$

We begin by observing that $\mathcal{J}$ is coercive in the weakly[3] closed set $A$. Indeed, from the Poincaré inequality (see Proposition A.3.12), for any $v \in A$ we have

$$\begin{aligned}\int v^2 = \int [(v - u_0) + u_0]^2 &\leq 2 \int (v - u_0)^2 + 2 \int u_0^2 \\ &\leq 2C_1 \int |\nabla(v - u_0)|^2 + 2 \int u_0^2 \\ &\leq 4C_1 \int |\nabla v|^2 + 4C_1 \int |\nabla u_0|^2 + 2 \int u_0^2,\end{aligned}$$

i.e.

$$\int |\nabla v|^2 \geq C_2 \int v^2 - C_3,$$

where $C_1, C_2, \ldots$ denote different positive constants. From the above inequality one can easily deduce the coerciveness of $\mathcal{J}$ in $A$.

The proof of the semicontinuity of $\mathcal{J}$ is based on the convexity of the square function $|\xi|^2$, $\xi \in \mathbb{R}^N$. Indeed, this means that

$$|\xi|^2 \geq |\xi_0|^2 + 2\xi_0 \cdot (\xi - \xi_0), \ \forall \xi, \xi_0 \in \mathbb{R}^N.$$

Taking $\xi = \nabla w$ and $\xi_0 = \nabla u$, with $v, w \in H^1(\Omega)$, it follows that

$$\mathcal{J}(w) \geq \mathcal{J}(v) + 2 \int \nabla v \cdot (\nabla w - \nabla v), \tag{1.3}$$

for every $v, w \in H^1(\Omega)$ and thus $\mathcal{J}$ is convex. In particular, if $w = v_n$ is weakly convergent to $v$ in $H^1(\Omega)$, we have

$$\lim_{n \to +\infty} \int \nabla v \cdot (\nabla v_n - \nabla v) = 0$$

and therefore

$$\liminf_{n \to +\infty} \mathcal{J}(v_n) \geq \mathcal{J}(v).$$

The uniqueness is due to the strict convexity[4] of $\mathcal{J}$ (which is also due to the strict convexity of $|\xi|^2$).

---

[3] Note that $A$ is clearly convex, thus it is sufficient to observe that $A$ is closed in the topology of the norm.

[4] That is, the strict inequality in (1.3) is satisfied for every $v \neq w \in H^1(\Omega)$.

To prove (ii), it suffices to note that, for every fixed $v \in H_0^1(\Omega)$, the function $u + tv \in A$, for every $t \in \mathbb{R}$, and the real function

$$\varphi(t) = \mathcal{J}(u + tv), \ t \in \mathbb{R}$$

has a minimum at $t = 0$. Therefore,

$$\begin{aligned} 0 = \varphi'(0) &= \lim_{t\to 0} \frac{\mathcal{J}(u+tv) - \mathcal{J}(u)}{t} \\ &= \lim_{t\to 0} \frac{2t\int \nabla u \cdot \nabla v + t^2 \int |\nabla v|^2}{t} \\ &= 2\int \nabla u \cdot \nabla v. \end{aligned}$$

□

The next application will be about the b.v.p.

$$\begin{aligned} -\Delta u &= h, \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.4}$$

with $h \in L^2(\Omega)$. In this case the Euler functional is $\mathcal{J} : H_0^1(\Omega) \longrightarrow \mathbb{R}$ given by

$$\mathcal{J}(v) = \frac{1}{2}\int |\nabla v|^2 - \int h\, v, \quad v \in H_0^1(\Omega). \tag{1.5}$$

**Corollary 1.2.9** *For every fixed $h \in L^2(\Omega)$, consider the functional $\mathcal{J}$ defined in $H_0^1(\Omega)$ by (1.5). Then there exists a unique $u \in H_0^1(\Omega)$ satisfying*

$$\mathcal{J}(u) = \min_{v \in H_0^1(\Omega)} \mathcal{J}(v).$$

*In particular, $u$ is a weak solution for the b.v.p. (1.4).*

*Proof* Similar arguments to the ones for the Dirichlet principle can be used to show the corollary. The details are left to the reader. □

*Remark 1.2.10* Above we have dealt with the Laplace operator for the sake of simplicity, only. It can be substituted by any second order uniformly elliptic operator like

$$-\sum \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u,$$

where $a_{ij}(x)$ and $c(x)$ are bounded and measurable on $\overline{\Omega}$, $c(x) \ge 0$ in $\overline{\Omega}$ and $a_{ij}(x) = a_{ji}(x)$ is uniformly elliptic, namely

$$\exists \alpha > 0 : \sum a_{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2, \ \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^N.$$

### *1.2.4 Regularity of the Solutions*

The regularity of weak solutions of (1.4) are stated in the following theorem.

**Theorem 1.2.11** *1. If $\partial\Omega$ is of class $C^{1,1}$ and $h \in L^p(\Omega)$ for some $p \in [2, +\infty)$ then the unique weak solution $u \in H_0^1(\Omega)$ of (1.4) belongs to $W^{2,p}(\Omega)$ and*

$$\|u\|_{W^{2,p}} \le C\,\|h\|_{L^p},$$

*for some $C > 0$. In particular, if $h \in C(\overline{\Omega})$ then $u \in C^1(\overline{\Omega})$.*
*2. If $\partial\Omega$ is of class $C^{2,\nu}$, $0 < \nu < 1$, and $h \in C^{0,\nu}(\overline{\Omega})$, then $u \in C^{2,\nu}(\overline{\Omega})$ is a classical solution of (1.4) and*

$$\|u\|_{C^{2,\nu}} \le C\,\|h\|_{C^{0,\nu}},$$

*for some $C > 0$.* □

The former inequalities are known as Agmon–Douglis–Nirenberg estimates or $L^p$-theory. The latter ones are the Schauder estimates.

### *1.2.5 The Inverse of the Laplace Operator*

For an open bounded set $\Omega$ in $\mathbb{R}^N$, we can define a linear operator $K : L^2(\Omega) \longrightarrow H_0^1(\Omega)$ by setting $K(h) = u$, the solution of (1.4). By the compact embedding Theorem 1.1.3, $K$ is compact as a map from $L^2(\Omega)$ into itself. In addition, the restriction of $K$ to $H_0^1(\Omega)$ into itself is also compact (see Exercise 5).

Similarly, we can use the Schauder estimate given in Theorem 1.2.11 to consider, for instance, $K$ as a map from $C^{0,\nu}(\overline{\Omega})$ into itself. The Ascoli compactness theorem implies that $K$ is also compact (see Exercise 6). Moreover, using the last statement in Theorem 1.2.11-1, it can be verified that $K$ is compact as a map from $C(\overline{\Omega})$ into itself.

## 1.3 Linear Elliptic Eigenvalue Problems

Let $\Omega \subset \mathbb{R}^N$ be an open and bounded subset, $r \in \left(\frac{N}{2}, \infty\right) \cap (1, \infty)$ and $m \in L^r(\Omega)$ a function (weight). We consider the weighted eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda m(x) u, \quad & x \in \Omega \\ u &= 0, & x \in \partial\Omega. \end{aligned} \tag{1.6}$$

That is, we look for pairs $(\lambda, u) \in \mathbb{R} \times (H_0^1(\Omega) \setminus \{0\})$ such that (1.6) holds in the weak sense, i.e.,

$$\int \nabla u \cdot \nabla v\, dx = \lambda \int m u v\, dx, \quad \forall v \in H_0^1(\Omega). \tag{1.7}$$

In this case, we say that $\lambda$ is an eigenvalue and $u$ an associated eigenfunction.

By the Sobolev embedding theorem (see Theorem A.4.3), we have

$$H_0^1(\Omega) \hookrightarrow \begin{cases} L^{2N/(N-2)}(\Omega), & \text{if } N \geq 3 \\ L^t(\Omega) \ \ (\forall t \geq 1), & \text{if } N \leq 2. \end{cases}$$

Since $m \in L^r(\Omega)$ we observe then that

$$mu \in \begin{cases} L^{2N/(N+2)}(\Omega), & \text{if } N \geq 3 \\ L^t(\Omega) \ \ (\forall t \in (1,r)) & \text{if } N \leq 2, \end{cases}$$

and thus the right-hand side of (1.7) is well defined.

Clearly, $\lambda = 0$ is not an eigenvalue of (1.6). Hence, we devote our attention to look for nonzero eigenvalues of this problem. In order to do so, we consider $H \equiv H_0^1(\Omega)$ and for a fixed number $t_0$ in $(1,r)$ we pick

$$p = \begin{cases} \frac{2N}{N-2}, & \text{if } N \geq 3 \\ t_0, & \text{if } N \leq 2. \end{cases}$$

Given $f \in L^p(\Omega)$, let $w = Kf$ be the unique (weak) solution of the problem

$$\begin{aligned} -\Delta w &= f, \quad x \in \Omega \\ w &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Note that, in this way, the operator $K : L^p(\Omega) \to H$ is linear and continuous. We define also the operator $T : H \to H$ by $Tu = K(mu)$ for every $u \in H$, i.e., $Tu$ is the unique point in $H$ satisfying

$$\int \nabla Tu \cdot \nabla v = \int muv, \ \ \forall v \in H. \tag{1.8}$$

It is easy to verify that $T$ is linear and symmetric (i.e., $(Tu, v) = (u, Tv)$, for every $u, v \in H$).

## *1.3.1 Linear Compact Operators*

In this subsection we give a short survey on linear compact operators, which will play a fundamental role in dealing with elliptic boundary value problems. For more details and proofs we refer to [36]. The following definition has already been given Section 1.1.1.

**Definition 1.3.1** If $X$ and $Y$ are Banach spaces, an operator $T : X \longrightarrow Y$ is compact if it is continuous and $T(A)$ is relatively compact for all bounded sets $A \subset X$.

Let us remark that the composition of a continuous operator with a compact operator is also a compact operator (see Exercise 4). For example, the operator $T$ given by (1.8) is compact. Indeed, if $\{u_n\} \subset H$ is weakly converging to a $u \in H$, by using the compact embedding of $H$ into $L^t(\Omega)$ for $t \in [1, 2^*)$, we deduce that this sequence is strongly converging in $L^t(\Omega)$ and, hence, applying the Hölder inequality, we obtain

$$\{mu_n\} \longrightarrow mu \mathrm{in} L^p(\Omega)$$

from which, the continuity of $K$ implies that $\{Tu_n\}$ is strongly converging in $H$ to $Tu$ and $T$ is compact. For compact operators, the Fredholm alternative applies yielding the following.

**Theorem 1.3.2** *Let $X$ be a Banach space and let $T : X \longrightarrow X$ be linear and compact. Then:*

1. *$\mathrm{Ker}\,[I - T]$ is finite dimensional;*
2. *$\mathrm{Range}\,[I - T]$ is closed, has finite codimension and $\mathrm{Range}\,[I - T] = \mathrm{Ker}\,[I - T^*]^{\perp}$, where $T^*$ denotes the adjoint of $T$;*
3. *$\mathrm{Ker}\,[I - T] = \{0\} \Leftrightarrow \mathrm{Range}\,[I - T] = X$.* □

*Remark 1.3.3* A linear operator $L : X \to X$ is called a Fredholm operator if $\dim \mathrm{Ker}\, L < \infty$ and $\mathrm{Range}\, L$ is closed and has finite codimension. In this case, the index of $L$ is $\dim \mathrm{Ker}\, L - \mathrm{codim}\, \mathrm{Range}\, L$. In particular, the preceding theorem states that $I - T$ is a Fredholm operator of zero index.

Let $T$ be compact and set $A_\gamma(u) = T(u) - \gamma u$.

**Definition 1.3.4** The resolvent of $T$ is the set

$$\rho(T) = \{\gamma \in \mathbb{R} : A_\gamma \text{ is bijective from } X \text{ to itself}\}$$

The spectrum $\sigma(T)$ of $T$ is defined as $\sigma(T) = \mathbb{R} \setminus \rho(T)$. $A$ $\gamma \in \mathbb{R}$ such that $\mathrm{Ker}\,[A_\gamma] \neq \{0\}$ is called an eigenvalue of $T$ and $\mathrm{Ker}\,[A_\gamma]$ is called eigenspace associated to the eigenvalue $\gamma$. We also say that $\mu \in \mathbb{R}$ is a characteristic value of $T$ if the kernel of the operator $u \mapsto \mu T(u) - u$ is different from $\{0\}$.

*Remark 1.3.5* If $\gamma \notin \sigma(T)$ then the closed graph theorem implies that $A_\gamma$ is invertible and has a continuous inverse [36, Corollary 2.7].

Concerning the spectrum of $T$, the Riesz–Fredholm theory provides the following result.

**Theorem 1.3.6** *Let $T$ be linear and compact. Then $\sigma(T)$ is compact and $\sigma(T) \subset [-\|T\|, \|T\|]$. Furthermore, if $X$ is infinite dimensional, one has:*

1. *$0 \in \sigma(T)$;*
2. *Every $\gamma \in \sigma(T) \setminus \{0\}$ is an eigenvalue of $T$;*
3. *Either $\sigma(T) = \{0\}$, or $\sigma(T)$ is finite, or $\sigma(T) \setminus \{0\}$ is a sequence which tends to 0.*

*Moreover, for every $\gamma \in \sigma(T) \setminus \{0\}$, there exists $m \geq 1$ such that*

$$\mathrm{Ker}\,[A_\gamma^k] = \mathrm{Ker}\,[A_\gamma^{k+1}], \quad \forall\, k \geq m, \tag{1.9}$$

*and there holds*

$$\text{Range}[A_\gamma^k] = \text{Range}[A_\gamma^{k+1}], \quad X = \text{Ker}\,[A_\gamma^m] \oplus \text{Range}[A_\gamma^m]. \qquad \square$$

**Definition 1.3.7** The multiplicity of an eigenvalue $\gamma$ of $T$ is, by definition, the least integer $m$ such that (1.9) holds. When $m = 1$ we say that the eigenvalue is simple.

### *1.3.2 Variational Characterization of The Eigenvalues*

Let T be defined by (1.8). Then $\lambda \in \mathbb{R} \setminus \{0\}$ is an eigenvalue of (1.6) if and only if $\lambda$ is a characteristic value of $T$. Applying Theorem 1.3.6, we deduce the following result (see also [47]).

**Theorem 1.3.8** *Assume that $\Omega \subset \mathbb{R}^N$ is bounded and open, $r \in \left(\frac{N}{2}, \infty\right) \cap (1, \infty)$, and $m \in L^r(\Omega)$. Consider the sets $\Omega_+ = \{x \in \Omega : m(x) > 0\}$ and $\Omega_- = \{x \in \Omega : m(x) < 0\}$. The following assertions hold.*

*(i)* *0 is not an eigenvalue of (1.6).*

*(ii) (a) If the Lebesgue measure $|\Omega_+|$ of $\Omega_+$ is zero, then (1.6) has no positive eigenvalue.*

*(b) If $|\Omega_+| > 0$, then the positive eigenvalues of (1.6) define a nondecreasing unbounded sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$. In addition, $\lambda_n$ is characterized by*

$$\frac{1}{\lambda_n} = \sup_{F\in\mathcal{F}_n} \inf \left\{ \int m(x)u^2(x)dx \,:\, \int |\nabla u(x)|^2\, dx = 1, u \in F \right\}$$

*where $\mathcal{F}_n = \{F \subset H \,:\, F \text{ is a subspace with dim } F = n\}$.*

*(iii) (a) If $|\Omega_-| = 0$, then (1.6) has no negative eigenvalue.*

*(b) If $|\Omega_-| > 0$, then the negative eigenvalues of (1.6) define a nonincreasing unbounded sequence $\{\lambda_{-n}\}_{n\in\mathbb{N}} \subset (-\infty, 0)$. In addition, $\lambda_{-n}$ is characterized by*

$$\frac{1}{\lambda_{-n}} = \inf_{F\in\mathcal{F}_n} \sup \left\{ \int m(x)u^2(x)dx \,:\, \int |\nabla u(x)|^2\, dx = 1, u \in F \right\}. \qquad \square$$

Remark that the eigenvalues $\lambda_n$ have associated eigenfunctions $u_n \in H_0^1(\Omega)$. However, by the regularity results, if $\partial\Omega$ and $m$ are smooth, then $u_n \in C^2(\overline{\Omega})$ and, hence, they are eigenfunctions in a classical sense.

If $m \equiv 1$, then $\Omega = \Omega_+$ and $1/\lambda_1 = \sup\{\int u^2 \,:\, \int |\nabla u|^2 = 1\}$, and consequently $\lambda_1$ is the best constant in the Poincaré inequality (1.1) for $p = 2$.

**Corollary 1.3.9** (Best constant for the Poincaré inequality) *If $\Omega \subset \mathbb{R}^N$ is open and bounded, then*

$$\lambda_1 = \min \left\{ \int |\nabla u|^2 \,:\, u \in H_0^1(\Omega),\ \int u^2 = 1 \right\}. \qquad \square$$

*Remark 1.3.10* Every minimizer $\varphi$ is an eigenfunction.

It is possible to prove that the eigenvalues of (1.6) are continuously depending on the weight $m$. For simplicity, we assume that $m > 0$.

**Proposition 1.3.11** *Assume that $\Omega \subset \mathbb{R}^N$ is open and bounded, $r \in \left(\frac{N}{2}, \infty\right) \cap (1, \infty)$ and $m \in L^r(\Omega)$.*

*(i) If $\overline{m} \in L^r(\Omega)$ satisfies $m(x) \le \overline{m}(x)$ a.e. $x \in \Omega$, then for all $j \ge 1$ there holds*

$$\lambda_j(m) \ge \lambda_j(\overline{m})$$

*with strict inequality provided that, in addition, $|\{x \in \Omega : m(x) < \overline{m}(x)\}| > 0$.*

*(ii) If $m_n \in L^r(\Omega)$, $m_n > 0$, is converging in $L^{t_1}(\Omega)$ to $m$ with*

$$t_1 = \begin{cases} N/2, & \text{if } N \ge 3 \\ \in \left(\max\{\frac{N}{2}, 1\}, r\right], & \text{if } N = 2, \end{cases}$$

*then for all $j \ge 1$ there holds*

$$\lim_{n \to +\infty} \lambda_j(m_n) = \lambda_j(m).$$ □

One of the main properties of the first positive and negative eigenvalues $\lambda_1, \lambda_{-1}$ of (1.6) is that they are simple and the associated eigenfunctions have a sign.

**Theorem 1.3.12** (Simplicity of the first eigenvalues) *Assume that $\Omega \subset \mathbb{R}^N$ is open and bounded, $r \in \left(\frac{N}{2}, \infty\right) \cap (1, \infty)$ and $m \in L^r(\Omega)$.*

*(i) If $|\Omega_+| > 0$ then the first positive eigenvalue $\lambda_1$ of (1.6) is simple (with one algebraic and geometric multiplicity) and its associated eigenspace is spanned by an eigenfunction $\phi_1 \in H_0^1(\Omega)$ such that $\phi_1(x) > 0$ a.e. $x \in \Omega$. In addition, $\lambda_1$ is the unique positive eigenvalue having an associated eigenfunction which does not change sign.*

*(ii) If $|\Omega_-| > 0$ then the first negative eigenvalue $\lambda_{-1}$ of (1.6) is simple and its associated eigenspace is spanned by an eigenfunction $\phi_{-1} \in H_0^1(\Omega)$ such that $\phi_{-1}(x) > 0$ a.e. $x \in \Omega$. In addition, $\lambda_{-1}$ is the unique negative eigenvalue having an associated eigenfunction which does not change sign.* □

Corollary 1.3.9, Proposition 1.3.11 and Theorem 1.3.12 are closely related to the maximum principle. First, we give the result for the classical formulation.

**Theorem 1.3.13** (Maximum principle) *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$\begin{cases} -\Delta u \ge \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

*If $\lambda < \lambda_1$ then $u \ge 0$ in $\Omega$. Moreover, either $u > 0$ in $\Omega$ or $u \equiv 0$ in $\Omega$.* □

Similarly, we have the following result for the weak formulation of the problem.

**Theorem 1.3.14** *Assume that* $m \in L^\infty(\Omega)$ *with* $m^+ := \max\{m, 0\} \not\equiv 0$, $0 \le h \in L^{2N/(N+2)}(\Omega)$ *and* $u \in H_0^1(\Omega)$ *is a solution of the problem*

$$\begin{cases} -\Delta u = \lambda m u + h, & \text{in } \Omega \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases}$$

*If* $\lambda < \lambda_1$ *then* $u \ge 0$ *(a.e.) in* $\Omega$. *Moreover, if* $h > 0$ *in a set of positive measure, then* $u > 0$ *in* $\Omega$. □

# Chapter 2
# Some Fixed Point Theorems

In this chapter we discuss the classical Banach contraction principle and a fixed point theorem for increasing operators that will be used in connection to sub- and super-solutions of elliptic boundary value problems.

## 2.1 The Banach Contraction Principle

Let $X$ be a complete metric space. An operator $T : X \to X$ is a *contraction* if there exists $\alpha \in (0, 1)$ such that

$$d_X(T(u), T(v)) \leq \alpha\, d_X(u, v), \quad \forall\, u, v \in X, \tag{2.1}$$

where $d_X(u, v)$ denotes the distance from $u$ to $v$ in $X$.

*Remark 2.1.1* From (2.1) it immediately follows that $T$ is continuous.

**Theorem 2.1.2** *If $X$ is a complete metric space and $T$ is a contraction on $X$ which maps $X$ into itself, then there exists a unique $z \in X$ such that $T(z) = z$.*

*Proof* *Existence*. For any fixed $u_0 \in X$ let us define the sequence $u_k$ by setting

$$u_{k+1} = T(u_k), \quad k \in \mathbb{N}.$$

One has that for every $j \geq 1$

$$d_X(u_{j+1}, u_j) = d_X(T(u_j), T(u_{j-1})) \leq \alpha d_X(u_j, u_{j-1})$$

and this, by induction, implies

$$d_X(u_{j+1}, u_j) \leq \alpha^j d_X(u_1, u_0).$$

Then, it follows that

$$d_X(u_{k+1}, u_h) \leq \sum_{j=h}^{k} d_X(u_{j+1}, u_j) \leq \left[\sum_{j=h}^{k} \alpha^j\right] d_X(u_1, u_0).$$

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82,
DOI 10.1007/978-0-8176-8114-2_2, © Springer Science+Business Media, LLC 2011

Since $0 < \alpha < 1$, $u_k$ is a Cauchy sequence. Let $z \in X$ be such that $u_k \to z$. Passing to the limit into $u_{k+1} = T(u_k)$ and using the fact that $T$ is continuous, it follows that $z = T(z)$.

*Uniqueness*. Let $z_1, z_2 \in X$ be fixed points of $T$. From this and (2.1) we infer

$$d_X(z_1, z_2) = d_X(T(z_1), T(z_2)) \leq \alpha d_X(z_1, z_2).$$

Since $\alpha < 1$, it follows that $z_1 = z_2$. □

As a typical application of the Banach contraction principle we can prove the existence and uniqueness of solutions of the Cauchy problem for a first order differential equation. This will be achieved by transforming the differential problem into an equivalent integral equation.

Let $(x_0, y_0)$ be a point in a domain $\Omega \subset \mathbb{R}^2$. For a continuous function $f : \Omega \longrightarrow \mathbb{R}$, we consider the Cauchy problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0. \end{cases} \tag{2.2}$$

By a (local) solution of (2.2) we mean a $C^1$ function $y(x)$ defined in some interval $(a, b) \subset \mathbb{R}$ such that $(x, y(x)) \in \Omega$ and $y'(x) = f(x, y(x))$ for every $x \in (a, b)$ which passes by the point $(x_0, y_0)$, i.e. $y(x_0) = y_0$.

**Lemma 2.1.3** *The Cauchy problem* (2.2) *is equivalent to the integral equation*

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{2.3}$$

*Proof* If $y(x)$ satisfies (2.3) then, clearly, $y(x_0) = y_0$. Moreover, differentiating one finds

$$y'(x) = f(x, y(x)).$$

Hence $y(x)$ is a solution of (2.2). Conversely, let $y(x)$ be a solution of (2.2). Integrating from $x_0$ to $x$ the identity $y'(x) \equiv f(x, y(x))$ we get

$$\int_{x_0}^{x} y'(t)dt = \int_{x_0}^{x} f(t, y(t))dt.$$

Using the initial condition $y(x_0) = y_0$ we deduce (2.3). □

**Definition 2.1.4** We say that $f(x, y)$ is locally Lipschitzian with respect to $y$ at $(x_0, y_0)$ if there exist a neighborhood $U$ of $(x_0, y_0)$ and $L > 0$ such that

$$|f(x, y) - f(x, y_1)| \leq L\, |y - y_1|, \quad \forall\ (x, y), (x, y_1) \in U. \tag{2.4}$$

If the preceding relationship is valid in all the domain $\Omega$ of $f$ we say that $f$ is (globally) Lipschitzian on $\Omega$ with respect to $y$.

Obviously, any function $f$ which is $C^1$ with respect to $y$ in $\Omega$ is locally Lipschitzian on $\Omega$ with respect to $y$. On the other hand, any Lipschitzian function with respect to $y$ is continuous in the variable $y$. But the converse is not true. For example, $f(x, y) = \sqrt{|y|}$ is not Lipschitzian at $(0, 0)$.

**Theorem 2.1.5** *Suppose that $f(x, y)$ is continuous and locally Lipschitzian with respect to $y$ at $(x_0, y_0)$. Then the Cauchy problem (2.2) has a unique solution $y(x)$ defined in a neighborhood of $x_0$.*

*Proof* Let $I = [x_0 - \delta, x_0 + \delta]$ with

$$0 < \delta < \min \left\{ \frac{1}{\mathrm{L}}, \frac{\mathrm{a}}{\mathrm{M}} \right\},$$

where $a, L > 0$ are chosen in such a way that (2.4) holds in $U = [x_0 - a, x_0 + a] \times [y_0 - a, y_0 + a]$ and $M = \sup_{(x,y) \in U} |f(x, y)|$. We will use the Banach contraction principle to show that the equivalent integral Eq. (2.3) has a unique solution in $I$. Let also denote by $X$ the Banach space $C(I)$ endowed with the sup norm

$$||y|| = \sup_{x \in I} |y(x)|$$

and consider the ball $B$ in $X$ of radius $a$ centered at $y_0$, that is,

$$B = \{y \in X : ||y - y_0|| \leq a\}.$$

Define the operator $T : X \mapsto X$ by setting

$$T[y](x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{2.5}$$

First of all, let us show that $T(B) \subset B$. Actually,

$$|T[y](x) - y_0| \leq M\,\delta < a.$$

Taking the supremum in $I$, we find $||T[y] - y_0|| < a$ and hence $T[y] \in B$. Next, we show that $T$ is a contraction on $B$. Actually, using the fact that $f$ is locally Lipschitzian we get

$$\begin{aligned} |T[y](x) - T[y_1](x)| &\leq \int_{x_0}^{x} |f(t, y(t)) - f(t, y_1(t))|dt \\ &\leq \int_{x_0}^{x} L\,|y(t) - y_1(t))|dt \leq \delta L\,||y - y_1||. \end{aligned}$$

Taking again the supremum in $I$,

$$||T[y] - T[y_1]|| \leq \delta L\,||y - y_1||.$$

Since $\delta L < 1$, $T$ is a contraction. Using the Banach contraction principle, we infer that $T$ has a unique fixed point $y^*$ on $B$. From $T[y^*] = y^*$ we deduce

$$y^*(x) = T[y^*](x) = y_0 + \int_{x_0}^{x} f(t, y^*(t))dt.$$

Therefore $y^*$ is the (unique) solution of (2.3) we were looking for. □

*Remark 2.1.6* Observe that the existence result proved above is local. Indeed, the interval of existence $I = [x_0 - \delta, x_0 + \delta]$ depends on $L$, $M$, and on the initial condition. The following example shows that the local result is the only one we can hope for. Consider the Cauchy problem

$$\begin{cases} y' = y^2, \\ y(0) = p > 0. \end{cases}$$

One checks that

$$y(x) = \frac{p}{1 - px}$$

satisfies the Cauchy problem. The maximal interval of definition of this solution is $(0, p^{-1})$ and depends on the initial condition. Let us point out that $f(y) = y^2$ is not globally Lipschitzian.

*Remark 2.1.7* If $\Omega$ is a strip $\Omega = \{(x, y) : a < x < b,\ y \in \mathbb{R}\}$ and $f$ is globally Lipschitzian on this strip, then (2.2) has a unique solution defined on all $(a, b)$ ($a$ can be $-\infty$ and/or $b$ can be $+\infty$).

*Remark 2.1.8* If $f$ is not Lipschitzian, but is merely continuous, it is possible to prove that (2.2) has a solution, defined locally near $x_0$ (Peano's theorem, see Exercise 18), though the uniqueness can fail. For example, the problem

$$\begin{cases} y' = \sqrt{|y|}, \\ y(0) = 0, \end{cases}$$

has infinitely many solutions: one is $y \equiv 0$; in addition for any $a > 0$ any function

$$y(x) = \begin{cases} 0, & \text{for } |x| < a, \\ \frac{1}{4}\,(x - a)|x - a|, & \text{for } |x| \geq a, \end{cases}$$

is also a solution.

In the next chapter, as a second application of the Banach contraction principle, we will prove the local inversion theorem (see Theorem 3.1.1).

## 2.2 Increasing Operators

In this section we will discuss another iteration scheme on ordered Banach spaces.

Let $X$ be a Banach space endowed with an *ordering* $\leq$ such that (*linear ordering*)

$$v \leq w \Rightarrow \alpha v + z \leq \alpha w + z, \quad \forall\, v, w, z \in X, \quad \forall \alpha \geq 0.$$

We write $w \geq v$ if and only if $v \leq w$. We will also suppose that the norm in $X$ is related to the ordering by the fact that there exists $C > 0$ such that

$$0 \leq v \leq w \Rightarrow \|v\| \leq C\|w\|. \tag{2.6}$$

We say that an operator $T : X \to X$ is increasing if

$$v \le w \Rightarrow T(v) \le T(w), \quad \forall\, v, w \in X.$$

If $v \in X$ satisfies $v \le T(v)$, it is called a sub-solution of the fixed point equation of $T$, $T(u) = u$. Similarly, $w \in X$ is a super-solution if $T(w) \le w$.

Given a sub-solution $v \in X$, we define an iteration scheme by setting

$$\begin{cases} u_0 = v \\ u_{k+1} = T(u_k), \quad k = 1, 2, \dots \end{cases} \tag{2.7}$$

**Lemma 2.2.1** *Let $T : X \to X$ be an increasing operator and suppose that there exist a sub-solution $v \in X$ and a super-solution $w \in X$ of the fixed point equation of $T$ such that $v \le w$. Then the sequence $u_k$ given by (2.7) satisfies $v \le u_k \le u_{k+1} \le w$, for all $k = 0, 1, \dots$.*

*Proof* We argue by induction. By the definition of sub-solution, for $k = 0$ one has $u_1 = T(u_0) = T(v) \ge v$. Moreover, from $u_k \ge u_{k-1}$ and the fact that $T$ is increasing we infer that $T(u_k) \ge T(u_{k-1})$ and hence

$$u_{k+1} = T(u_k) \ge T(u_{k-1}) = u_k.$$

Similarly, one has that $u_0 = v \le w$ and, if $u_k \le w$, the fact that $T$ is increasing and the definition of super-solution yield $u_{k+1} = T(u_k) \le T(w) \le w$. □

**Theorem 2.2.2** *Let $T \in C(X, X)$ be compact and increasing and assume that there exist a sub-solution $v \in X$ and a super-solution $w \in X$ of the fixed point equation of $T$ satisfying $v \le w$. Then the sequence $u_k$ given by (2.7) converges to some $u \in X$ such that $T(u) = u$. Moreover, $v \le u \le w$.*

*Proof* Since, by Lemma 2.2.1, $0 \le u_k - v \le w - v$, the property (2.6) implies that

$$\|u_k\| \le \|u_k - v\| + \|v\| \le C\|w - v\| + \|v\| \le C_1.$$

Since $T$ is a compact operator, the sequence $T(u_k)$ is relatively compact and, up to a subsequence, it converges to some $u \in X$ (actually by the monotonicity property of $u_k$, the whole sequence converges). From $u_{k+1} = T(u_k)$ and the continuity of $T$, we infer that $u = T(u)$. Moreover, again using Lemma 2.2.1, it follows that $v \le u \le w$. □

*Remark 2.2.3* By the definition of $u_k$, $u = \lim_{k\to\infty} u_k$ is the minimal fixed point of $T$ in $\{z \in X : v \le z \le w\}$.

Later on, Theorem 2.2.2 will be applied to the study of the existence of solutions of nonlinear elliptic boundary value problems via sub- and super-solutions (see Sect. 7.2).

# Chapter 3
# Local and Global Inversion Theorems

This chapter deals with the local inversion theorem and the implicit function theorem in Banach spaces. The Lyapunov–Schmidt reduction is discussed in Sect. 3.3. In Sect. 3.4 we prove the global inversion theorem, which goes back to Hadamard and Caccioppoli. Section 3.5 deals with a global inversion theorem in the presence of fold singularities.

## 3.1 The Local Inversion Theorem

Let $X, Y$ be Banach spaces and let $F : X \to Y$. In the study of the existence of pairs $(u, h)$ satisfying the equation $F(u) = h$, it may occur that a "trivial" solution $(u_0, h_0)$ (i.e., $F(u_0) = h_0$) is known. The local inversion theorem is a classical result that allows us to solve an equation $F(u) = h$ in a neighborhood of $(u_0, h_0)$.

**Theorem 3.1.1** *Let $u_0 \in X$ and $h_0 \in Y$ be such that $F(u_0) = h_0$ and suppose that there exists a neighborhood $U_0 \subset X$ of $u_0$ such that*

*(i) $F \in C^1(U_0, Y)$;*
*(ii) $dF(u_0)$ is invertible (as a linear map from $X$ to $Y$).*

*Then there exists a neighborhood $U \subset U_0$ of $u_0$ and a neighborhood $V \subset Y$ of $h_0$ such that the equation $F(u) = h$ has a unique solution in $U$, for all $h \in V$. Furthermore, denoting by $F^{-1} : V \to U$ the inverse of $F|_U$, one has that $F^{-1}$ is of class $C^1$ and there holds for every $u \in U$*

$$dF^{-1}(h) = [dF(u)]^{-1}, \quad \text{where } F(u) = h.$$

*Proof* Up to translations, we can assume that $u_0 = 0$ and $h_0 = 0$. In order to apply the Banach contraction principle (see Theorem 2.1.2), we let $L = dF(0)$ and consider the map $\widetilde{F} : U_0 \to U_0$ defined by setting

$$\widetilde{F}(u) = u - L^{-1}F(u).$$

With this notation, $u$ solves $F(u) = h$ if and only if $u$ satisfies

$$\widetilde{F}(u) + L^{-1}h = u,$$

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82,
DOI 10.1007/978-0-8176-8114-2_3, © Springer Science+Business Media, LLC 2011

that is, $u$ is a fixed point of $F_h(u) := \widetilde{F}(u) + L^{-1}h$. Let $r > 0$ be such that the closed ball $B_r = \{u \in X : \|u\| \le r\}$ is contained in $U_0$. Since $F$ is $C^1$ in $U_0$, then $dF(w) \to dF(0) = L$ as $w \to 0$. Therefore, given $\alpha \in (0,1)$, there exists $\delta \in (0,r)$ such that

$$\sup_{\|w\| \le \delta} \|I - L^{-1} \circ dF(w)\| \le \alpha,$$

($I$ denotes the identity map in $X$). From the mean value theorem we infer

$$\|\widetilde{F}(u) - \widetilde{F}(v)\| \le \sup_{\|w\| \le \delta} \|I - L^{-1} \circ dF(w)\| \, \|u - v\| \le \alpha \|u - v\|. \tag{3.1}$$

for every $u, v \in B_\delta$. From this and using $\widetilde{F}(0) = 0$, we also infer

$$\|F_h(u)\| = \|\widetilde{F}(u) + L^{-1}h\| \le \alpha\|u\| + \|L^{-1}h\|, \quad \forall\, u \in B_\delta.$$

Choosing $\varepsilon > 0$ such that $\|L^{-1}h\| < (1-\alpha)\delta$ provided $\|h\| < \varepsilon$, we obtain

$$\|F_h(u)\| \le \delta, \quad \forall\, u \in B_\delta, \quad \forall\, \|h\| < \varepsilon.$$

In conclusion, if $\|h\| < \varepsilon$, then $F_h$ maps $B_\delta$ into itself and is a contraction in $B_\delta$. By the Banach contraction principle, it follows that $F_h$ has a unique fixed point $z_h \in B_\delta$ such that $F(z_h) = h$.

To show that $F^{-1}$ is continuous, we set $u = F^{-1}(h)$, $v = F^{-1}(k)$, namely $\widetilde{F}(u) + L^{-1}h = u$ and $\widetilde{F}(v) + L^{-1}k = v$. Therefore,

$$\|u - v\| \le \|\widetilde{F}(u) - \widetilde{F}(v)\| + \|L^{-1}\| \, \|h - k\|\,,$$

and using (3.1) we infer

$$\|u - v\| \le \alpha\|u - v\| + \|L^{-1}\| \, \|h - k\|.$$

This implies that

$$\|u - v\| \le \frac{\|L^{-1}\|}{1 - \alpha} \, \|h - k\| \tag{3.2}$$

and proves that $F^{-1}$ is continuous (in fact, Lipschitzian). To complete the proof, we have to show that

$$F^{-1}(k) - F^{-1}(h) - [dF(u)]^{-1}[k - h] = o(\|k - h\|). \tag{3.3}$$

From the differentiability of $F$ we infer $F(v) - F(u) - dF(u)[v - u] = o(\|v - u\|)$, namely $k - h = dF(u)[v - u] + o(\|v - u\|)$. This implies that $[dF(u)]^{-1}[k - h] = v - u + o(\|v - u\|)$. Substituting into (3.3) and taking into account that $v = F^{-1}(k)$ and $u = F^{-1}(h)$, we get

$$F^{-1}(k) - F^{-1}(h) - [dF(u)]^{-1}[k - h] = o(\|v - u\|).$$

Finally we use (3.2) to infer that $o(\|v - u\|) = o(\|k - h\|)$, and this completes the proof. □

*Remark 3.1.2* If $F$ is of class $C^k$, $k \geq 1$, it is possible to show that $F^{-1}$ is also of class $C^k$.

*Remark 3.1.3* If $F$ is not $C^1$ but merely differentiable at $u$, then the assertion of Theorem 3.1.1 is not satisfied. Indeed, elementary examples with $X = Y = \mathbb{R}$ show that $F$ can fail to be locally injective (see Exercise 14). In addition if $X = Y$ has infinite dimension, one can exhibit cases in which $F$ is neither locally injective nor surjective (see Exercise 15).

## 3.2 The Implicit Function Theorem

The implicit function theorem deals with the solvability of an equation as $F(\lambda, u) = 0$, where $\lambda$ is a parameter. To simplify the notation, we will suppose that $\lambda \in \mathbb{R}$, although the more general case in which $\lambda \in \mathbb{R}^n$ is quite similar.

**Theorem 3.2.1** *Let $X, Y$ be Banach spaces and fix $(\lambda_0, u_0) \in \mathbb{R} \times X$. Assume that $F$ is a $C^1$ map from a neighborhood of $(\lambda_0, u_0)$ in $\mathbb{R} \times X$ into $Y$ such that $F(\lambda_0, u_0) = 0$ and suppose that $d_u F(\lambda_0, u_0)$ is invertible. Then there exist a neighborhood $\Lambda$ of $\lambda_0$ and a neighborhood $U$ of $u_0$, such that the equation $F(\lambda, u) = 0$ has a unique solution $u = u(\lambda) \in U$ for all $\lambda \in \Lambda$. The function $u(\lambda)$ is of class $C^1$, and the following holds:*

$$u'(\lambda_0) = -[d_u F(\lambda_0, u_0)]^{-1} d_\lambda F(\lambda_0, u_0). \tag{3.4}$$

*Proof* Let $\Lambda_0 \subset \mathbb{R}$ denote a neighborhood of $\lambda_0$ and $U_0 \subset X$ a neighborhood of $u_0$ such that $F \in C^1(\Lambda_0 \times U_0, Y)$. Let us consider the auxiliary function $S : \Lambda_0 \times U_0 \to \Lambda_0 \times Y$ defined by setting

$$S(\lambda, u) = (\lambda, F(\lambda, u)).$$

We want to apply the local inversion theorem (Theorem 3.1.1) to $S$ at $(\lambda_0, u_0)$. The derivative $dS(\lambda_0, u_0)$ is the map

$$(\alpha, v) \mapsto (\alpha, d_\lambda F(\lambda_0, u_0)\alpha + d_u F(\lambda_0, u_0)[v]).$$

Let us consider the equation $dS(\lambda_0, u_0)[\alpha, v] = (\beta, h)$. It is immediate to check that this equation has a unique solution given by

$$\alpha = \beta, \quad v = [d_u F(\lambda_0, u_0)]^{-1}(h - d_\lambda F(\lambda_0, u_0)\beta),$$

and this implies that $dS(\lambda_0, u_0)$ is invertible. A straight application of the local inversion theorem to $S(\lambda, u) = (\lambda, 0)$ yields a $C^1$ map $R$, defined in a neighborhood $\Lambda \times U$ of $(\lambda_0, 0)$, such that $S \circ R(\lambda, h) = (\lambda, h)$ for all $(\lambda_0, h) \in \Lambda \times U$. This means that the components $(R_1(\lambda, h), R_2(\lambda, h))$ of $R$ satisfy

$$R_1(\lambda, h) = \lambda, \quad F(\lambda, R_2(\lambda, h)) = h,$$

and hence $u(\lambda) := R_2(\lambda, 0)$ is the function we are looking for. In order to find $u'(\lambda)$, it suffices to remark that $F(\lambda, u(\lambda)) = 0$ for all $\lambda \in \Lambda$. Taking the derivative at $(\lambda_0, u_0)$

we get

$$d_\lambda F(\lambda_0, u_0) + [d_u F(\lambda_0, u_0)]u'(\lambda_0) = 0,$$

and hence (3.4) holds. This completes the proof. □

## 3.3 The Lyapunov–Schmidt Reduction

In the sequel we will frequently deal with an equation like

$$Lu + H(u) = \lambda u, \quad u \in E, \tag{3.5}$$

where for simplicity $E$ is a Hilbert space, $L : E \to E$ is linear continuous and $H \in C^1(E, E)$ is such that $H(0) = 0$, $H'(0) = 0$. Setting $F(\lambda, u) = Lu + H(u) - \lambda u$, one has that $F(\lambda, 0) \equiv 0$ for all $\lambda \in \mathbb{R}$. In order to apply the implicit function theorem, we calculate

$$d_u F(\lambda, 0)[v] = Lv + H'(0)[v] - \lambda v = Lv - \lambda v.$$

Therefore, the implicit function theorem applies provided $\lambda \notin \sigma(L)$, where $\sigma(L)$ denotes the spectrum of $L$ (see Definition 1.3.4). In this case, the trivial solution $u = 0$ is the unique solution of (3.5) in a neighborhood of zero. Otherwise, we are in the presence of a singularity and we can use a procedure that goes back to Lyapunov and Schmidt (see [66, 67, 81]). Roughly, one splits the equation $Lu + H(u) = \lambda u$ in a system of two equations, into which one equation can be uniquely solved, while the other one inherits the effects of the singularity.

Let us suppose that $\lambda^*$ is an eigenvalue of $L$ and let $Z = \mathrm{Ker}\,(L - \lambda I)$, where $I$ denotes the identity map in $E$. $Z$ is closed and there exists a closed subset $W \subset E$ such that $E = Z \oplus W$. Let $P : E \to Z$ denote the projection onto $Z$ and set $Pu = z$ and $w = u - Pu$. Let us point out that $Lu = Lw$. With this notation, (3.5) is equivalent to the system

$$Lw + PH(z + w) = \lambda w, \tag{3.6}$$

$$(I - P)H(z + w) = \lambda z. \tag{3.7}$$

The former is called the *auxiliary equation* and the latter the *bifurcation equation*.

**Lemma 3.3.1** *For all* $(\lambda, z) \in \mathbb{R} \times Z$*, the auxiliary equation* (3.6) *has a unique solution* $w = w(\lambda, z)$ *which is of class* $C^1$*. Moreover there holds:* $w(\lambda, 0) = 0$*,* $w_z(\lambda, 0) = 0$*, and the derivative* $w_\lambda := d_\lambda w(\lambda, 0)$ *of* $w$ *with respect to* $\lambda$ *is also zero.*

*Proof* Consider $\widetilde{F} : \mathbb{R} \times Z \times W \to W$, defined by $\widetilde{F}(\lambda, z, w) = Lw + PH(z + w) - \lambda w$. One finds that $\widetilde{F}_w(\lambda, 0, 0)$ is the restriction of $L - \lambda I$ to $W$, which is invertible. Therefore the implicit function theorem applies and yields $w(\lambda, z)$ with the stated properties. As for $w_\lambda$, it suffices to take the derivative with respect to $\lambda$ of (3.6). One finds that $w_\lambda$ satisfies $Lw_\lambda - \lambda w_\lambda = 0$ and hence $w_\lambda = 0$. The derivative $w_z(\lambda, 0)$ can be found by differentiating $Lw + PH(z + w) - \lambda w = 0$. Since $H'(0) = 0$, one finds

$Lw_z(\lambda,0) - \lambda w_z(\lambda,0) = 0$, and this, taking into account that $w_z(\lambda,0) \in W$, implies $w_z(\lambda,0) = 0$. □

*Remark 3.3.2* If $H \in C^k(E,E)$ then $w$ is of class $C^k$ too, and the derivatives $d_z^i w(\lambda,z), i = 1,\dots,k$, can be evaluated by differentiating $Lw+PH(z+w)-\lambda w = 0$ with respect to $z$.

*Remark 3.3.3* The Lyapunov–Schmidt reduction procedure can be carried out if $F \in C^k(\mathbb{R} \times X, Y)$, where $X$ and $Y$ are Banach spaces.

## 3.4 The Global Inversion Theorem

Let $F$ be a map between two metric spaces $X, Y$.

**Definition 3.4.1** We say that $F : X \to Y$ is proper if $F^{-1}(C) := \{u \in X : F(u) \in C\}$ is compact for all compact sets $C \subset Y$.

*Remark 3.4.2* Any proper $F$ maps closed sets into closed sets.

**Proposition 3.4.3** *Let $F \in C(X,Y)$ be proper and locally invertible in $X$. Then for every $v \in Y$ the set $F^{-1}(\{v\})$ is finite and its cardinality is locally constant.*

*Proof* Since $F$ is proper, $F^{-1}(\{v\})$ is compact. Moreover, since $F$ is locally invertible on $X$, $F^{-1}(\{v\})$ is discrete and therefore $F^{-1}(\{v\})$ is finite. Let $u_i$, $i = 1,...,k$, be such that $F(u_i) = v$. Since $F$ is locally invertible, there exist neighborhoods $U_i \subset X$ of $u_i$ and $V \subset Y$ of $v$ such that $F$ is a homeomorphism between $U_i$ and $V$. We want to show that there is a neighborhood $W \subset V$ such that the cardinality of $F^{-1}(\{w\})$ is $k$, for every $w \in W$. If not, there exists a sequence $v_n \in V$ with $v_n \to v$ and $z_n \notin \bigcup_{i=1,\dots,k} U_i$ such that $F(z_n) = v_n$. From the properness of $F$ we infer that $z_n$ converges, up to a subsequence, to some $z \in X$ and, by continuity, $F(z) = v$. Therefore $z \in \bigcup_{i=1,\dots,k} U_i$, a contradiction. □

The *singular points* of $F$, denoted by $\Sigma = \Sigma(F)$, make up the set of $u \in X$ where $F$ is not locally invertible. We also define

$$\Sigma_0 = F^{-1}(F(\Sigma)),\ X_0 = X \setminus \Sigma_0,\ Y_0 = Y \setminus F(\Sigma). \tag{3.8}$$

The following proposition extends Proposition 3.4.3 to maps with singularities.

**Proposition 3.4.4** *If $F \in C(X,Y)$ is proper, then the cardinality of $F^{-1}(\{v\})$ is locally constant on every connected component of $Y_0$.*

*Proof* It suffices to consider the restriction of $F$ to $X_0$ which is locally invertible on $X_0$ and is proper as a map from $X_0$ to $Y_0$. □

We are now in position to prove the following global inversion theorem.

**Theorem 3.4.5** *Suppose that $F \in C(X,Y)$ is proper and let us also assume that $X_0$ is arcwise connected and $Y_0$ is simply connected, where $X_0$ and $Y_0$ are given by*

(3.8). *Then* $F$ *is a homeomorphism from* $X_0$ *onto* $Y_0$. *In particular, if* $\Sigma = \emptyset$ *then* $F$ *is a global homeomorphism from* $X$ *onto* $Y$.

*Proof* The fact that $F(X_0) = Y_0$ follows immediately from Proposition 3.4.4. Let us prove the injectiveness. We will be brief, referring to [17, pp. 48–52] for a complete proof.

First, consider the square $Q = [0,1] \times [0,1]$ and take any $u \in X_0$. Let $S \in C(Q, Y_0)$ be a continuous surface such that $S(0,0) = v := F(u)$. It is possible to show that for every such $u$ and $S$ there exists a unique $R \in C(Q, X_0)$ such that $F \circ R = S$.

Next, suppose by contradiction that there exist $u_0, u_1 \in X_0$ and $v \in Y_0$ such that $F(u_i) = v$, $i = 0, 1$. By assumption $X_0$ is arcwise connected and hence there is a path $p \in C([0,1], X_0)$ such that $p(0) = u_0$ and $p(1) = u_1$. The image $q = F \circ p$ is a closed curve in $Y_0$ which is simply connected. Thus there exists a homotopy $h \in C(Q, Y_0)$ such that for all $(s,t) \in Q$ there holds

$$h(s,0) = q(s), \quad h(s,1) = v, \quad h(0,t) = h(1,t) = v.$$

From the previous step we can find a unique surface $R \in C(Q, X_0)$ such that $R(0,0) = u_0$ and $F \circ R = h$ on $Q$. It is easy to check that the following facts hold:

1. $R(1,0) = u_1$;
2. $F(R(0,t)) = h(0,t) = v$;
3. $F(R(s,t)) = h(s,1) = v$;
4. $F(R(1,t)) = h(1,t) = v$.

It follows that $R$ is constant on the set

$$(\{0\} \times [0,1]) \cup ([0,1] \times \{1\}) \cup (\{1\} \times [0,1])\,.$$

Then $R(1,0) = R(0,0) = u_0$, a contradiction with point 1. □

## 3.5 A Global Inversion Theorem with Singularities

In this section we will deal with a case in which the previous global inversion theorem does not apply. The first result in this direction has been given in [16].

Let $X, Y$ be Banach spaces and let $F \in C^2(X,Y)$. We set

$$\Sigma' = \{u \in X : dF(u) \text{ is not invertible}\}.$$

We shall suppose that $u \in \Sigma'$ is an *ordinary singular point*, namely it satisfies

(i) $\exists\, \phi = \phi_u \in X, \phi \neq 0$, such that $\mathrm{Ker}\,[dF(u)]) = \mathbb{R}\phi$:
(ii) $\mathrm{Range}\,[dF(u)])$ is closed and has codimension 1;
(iii) $d^2F(u)[\phi,\phi] \notin \mathrm{Range}\,[dF(u)])$.

The following theorem gives a precise geometric description of the range of $F$.

**Theorem 3.5.1** *Let $F \in C^2(X, Y)$ be proper and suppose that the singular set $\Sigma'$ is nonempty and connected and every $u \in \Sigma'$ is an ordinary singular point. Then $\Sigma$ is a connected $C^1$ manifold of codimension 1 in $X$.*

*Moreover, assume that $F(u) = v$ has a unique solution for every $v \in F(\Sigma')$. Then $F(\Sigma')$ is also a connected $C^1$ manifold of codimension 1 in $Y$ and there exist $Y_0, Y_2 \subset Y$, which are nonempty, open and connected, with the properties*

1. *$Y = Y_0 \cup Y_2 \cup F(\Sigma')$;*
2. *the equation $F(u) = v$ has no solution if $v \in Y_0$, a unique solution if $v \in F(\Sigma')$ and precisely two solutions if $v \in Y_2$.*

Above, by a manifold of codimension 1 in $X$ we mean that, locally, $\Sigma = G^{-1}(0)$ for some $G \in C^1(X, \mathbb{R})$ such that $dG(u) \neq 0$ for all $u \in \Sigma$.

The proof of this theorem will be carried out through several lemmas. In the sequel $v = F(u)$. First of all, since any $u \in \Sigma'$ is an ordinary singular point, there exist $W \subset X$ and $Z \subset Y$ such that $X = \mathbb{R}\phi \oplus W$ and $Y = Z \oplus \text{Range}\,[dF(u)]$. Moreover, we can choose $\psi \in Y^* \setminus \{0\}$ such that $\text{Range}\,[dF(u)] = \text{Ker}\,[\psi]$. Let $z \in Z$ be such that $\langle \psi, z \rangle = 1$ and let $P(v) = \langle \psi, v \rangle z$ denote the projection onto $Z$.

Next, given $\widehat{v} \in Y$ we look for $\widehat{u} \in X$ such that $dF(\widehat{u})[t\phi + w] = \widehat{v}$. Using the Lyapunov–Schmidt reduction, this equation is equivalent to the system

$$\begin{cases} P\,dF(\widehat{u})[t\phi + w] = P\widehat{v}. \\ (I - P)\,dF(\widehat{u})[t\phi + w] = (I - P)\widehat{v}. \end{cases} \tag{3.9}$$

**Lemma 3.5.2** *$\Sigma'$ is a $C^1$ manifold of codimension 1 in $X$.*

*Proof* The map $(I - P)\,dF(u)$ is invertible as a map from $W$ to $\text{Range}\,[dF(u)]$ and there exists $\varepsilon > 0$ such that $(I - P)\,dF(\widehat{u})$ is also invertible provided $\|\widehat{u} - u\| < \varepsilon$. Set $A = [(I - P)\,dF(\widehat{u})]^{-1}(I - P)$ in such a way that

$$w = A\widehat{v} - tA\,dF(\widehat{u})[\phi].$$

Then (3.9) becomes

$$\begin{cases} (i)\ \ tP\,dF(\widehat{u})[\phi] + P\,dF(\widehat{u})[A\widehat{v} - tA\,dF(\widehat{u})[\phi]] = P\widehat{v}. \\ (ii)\ w = A\widehat{v} - tA\,dF(\widehat{u})[\phi]. \end{cases} \tag{3.10}$$

If $P\,dF(\widehat{u})[\phi] - A\,dF(\widehat{u})[\phi] \neq 0$ system (3.10) has the unique solution given by

$$\widehat{t} = \frac{P\widehat{v} - P\,dF(\widehat{u})[A\widehat{v}]}{P\,dF(\widehat{u})[\phi] - A\,dF(\widehat{u})[\phi]}$$

$$\widehat{w} = A\widehat{v} - \widehat{t}A\,dF(\widehat{u})[\phi].$$

Therefore, $\widehat{u} \in \Sigma' \cap B_\varepsilon(u)$ whenever

$$G(\widehat{u}) := P\, dF(\widehat{u})[\phi] - A\, dF(\widehat{u})[\phi] = 0.$$

Since

$$dG(u)[\phi] = Pd^2F(u)[\phi, \phi] = \langle \psi, d^2F(u)[\phi, \phi] \rangle,$$

and $u$ is an ordinary singular point (cf. condition (iii)), then $dG(u)[\phi] \neq 0$ and $\Sigma'$ is a manifold of codimension 1 in $X$. □

**Lemma 3.5.3** *$F(\Sigma')$ is a connected $C^1$ manifold of codimension 1 in $Y$.*

*Proof* Let $u \in \Sigma'$. Consider the map, defined for $\widehat{u} \in X$ with $\|\widehat{u} - u\| \ll 1$, by setting $\Psi(\widehat{u}) = F(\widehat{u}) + G(\widehat{u})z$. It is easy to check that $d\Psi(u)$ is invertible and hence $\Psi$ is locally a diffeomorphism. Since $G(u) = 0$ for all $u \in \Sigma'$ it follows that, locally, $F(\Sigma') = \widetilde{G}^{-1}(0)$, where $\widetilde{G} = G \circ \Psi^{-1}$. In addition, $d\widetilde{G} \neq 0$ and the lemma follows. □

In the next lemma we suppose that $\langle \psi, d^2F(u)[\phi, \phi] \rangle > 0$. The case in which $\langle \psi, d^2F(u)[\phi, \phi] \rangle < 0$ requires obvious changes.

**Lemma 3.5.4** *Suppose that $\langle \psi, d^2F(u)[\phi, \phi] \rangle > 0$. Then there exist $\varepsilon, \delta > 0$ such that the equation $F(t\phi + w) = v + sz$, with $t\phi + w \in B_\varepsilon(u) := \{\widehat{u} \in X : \|\widehat{u} - u\| < \varepsilon\}$, has two solutions for all $0 < s < \delta$ and no solution for all $-\delta < s < 0$.*

*Proof* To simplify notation, we take $u = v = 0$ and consider the equation $F(t\phi + w) = sz$. Setting $L = dF(0)$ we get

$$Lw + \omega((t\phi + w) = sz, \quad \text{where } \omega(0) = 0,\ d\omega(0) = 0.$$

Using again the Lyapunov–Schmidt reduction, we find the system

$$\begin{cases} Lw + P\omega((t\phi + w) = 0, \\ (I - P)\omega((t\phi + w) = sz. \end{cases}$$

The first of the preceding equations can be handled by the implicit function theorem yielding a $w = w(t)$ of class $C^1$ such that $w(0) = 0$ and $dw(0) = 0$. Inserting $w$ in the second equation of the system we find

$$\chi(t) := \langle \psi, \omega(t\phi + w(t)) \rangle = s.$$

A straight calculation yields

$$\chi'(0) = 0, \quad \chi''(0) = \langle \psi, d^2F(0)[\phi, \phi] \rangle > 0$$

and the result follows. □

The local result stated in the previous lemma is completed by the following one.

**Lemma 3.5.5** *For any neighborhood $U$ of $u \in \Sigma'$ there exists a neighborhood $V$ of $v = F(u)$ such that $F^{-1}(V) \subset U$.*

*Proof* By contradiction, there exists a neighborhood $U^*$ of $u$ and a sequence $u_n \notin U^*$ such that $F(u_n) \to F(u)$. Using the properness of $F$, we find that, up to a subsequence, $u_n$ converges to some $u^* \notin U^*$. Moreover, $F(u^*) = F(u)$, a contradiction with respect to the assumption that the equation $F(u) = v$ has a unique solution for all $v \in F(\Sigma')$. □

*Proof of Theorem 3.5.1* The properties of $\Sigma'$ and $F(\Sigma')$ are proved in Lemmas 3.5.2 and 3.5.3. Since $F(\Sigma')$ is a (connected) manifold of codimension 1, it is possible to show that $Y \setminus F(\Sigma')$ has at most two connected components; see [17, p. 60]. Lemma 3.5.4 jointly with Lemma 3.5.5 imply that the equation $F(u) = v$ has zero or two solutions for $v \notin F(\Sigma')$. Therefore $Y \setminus F(\Sigma')$ has precisely two components proving statement 1. As for statement 2, it follows from the preceding discussion and the fact that the cardinality of $F^{-1}(v)$ is constant on every connected component of $Y \setminus F(\Sigma')$. □

# Chapter 4
# Leray–Schauder Topological Degree

To study the number of solutions of equations like

$$\Phi(u) = b,$$

where $\Omega$ is an open set in a Banach space $X$, $\Phi : \overline{\Omega} \longrightarrow X$ and $b \in X$, and based on a similar idea of Brouwer for continuous maps defined in finite-dimensional spaces, Leray and Schauder [64] introduced a topological tool, called the degree. It consists in assigning an integer number $d(\Phi, \Omega, b)$ with the property that the equation has at least one solution provided that $d(\Phi, \Omega, b) \neq 0$ (the existence property). In addition, it is desired to have an additivity of the degree, namely if the equation has only solutions in two disjoint open subsets $\Omega_1, \Omega_2$ of $\Omega$, then $\deg(\Phi, \Omega, b) = \deg(\Phi, \Omega_1, b) + \deg(\Phi, \Omega_2, b)$.

The topological character of the degree is due to the fact that, roughly speaking, it is possible to deduce the existence of a solution of $\Phi(u) = b$ by showing that the map can be continuously deformed (by a homotopy) on a map $\Phi_0$ for which the existence of a solution of the equation $\Phi_0(u) = b$ is known. We point out that this homotopy property is in general not satisfied for the function $\#(\Phi, \Omega, b)$, the *number of solutions of* $\Phi(u) = b$ *in* $\overline{\Omega}$. Indeed, it suffices to think of the example $\Omega = (-3, 3) \subset X = \mathbb{R}$, $b = 0$ and $\Phi_\lambda(x) = (x - \lambda)^3 - \lambda^2(x - \lambda)$. In this case, for $\lambda \in (0, 1]$ the equation $\Phi_\lambda(x) = 0$ has exactly three solutions $0, \lambda, 2\lambda$, while for $\lambda = 0$ there is just one solution, $x = 0$. This example shows that the definition of $d(\Phi, \Omega, b)$ requires some care.

In this chapter we discuss in detail the Leray–Schauder topological degree, which will be a fundamental tool for the applications to nonlinear problems in infinite-dimensional spaces. It is based on the the finite-dimensional Brouwer degree which, for the sake of brevity, is only sketched in an initial section.

## 4.1 The Brouwer Degree

The Brouwer degree is now a well known tool, discussed in several books, like [2, 15, 44, 71]. For this reason we will limit ourselves to give the definition of the Brouwer degree and to outline its main properties without proofs.

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82,
DOI 10.1007/978-0-8176-8114-2_4, 

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and consider a map $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and a point $b \in \mathbb{R}^N \setminus f(\partial\Omega)$. The above assumptions will always be understood in the sequel.

To define the Brouwer degree first we fix some notation. Let $f \in C(\overline{\Omega}, \mathbb{R}^N) \bigcap C^1(\Omega, \mathbb{R}^N)$. If the components of $f(x)$ are $f_i(x)$, we denote by $f'(x)$ the Jacobian of $f$ at $x$, namely the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x), & \frac{\partial f_1}{\partial x_2}(x), & \dots & \frac{\partial f_1}{\partial x_N}(x) \\ \frac{\partial f_2}{\partial x_1}(x), & \frac{\partial f_2}{\partial x_2}(x), & \dots & \frac{\partial f_2}{\partial x_N}(x) \\ & \dots & & \\ \frac{\partial f_N}{\partial x_1}(x), & \frac{\partial f_N}{\partial x_2}(x), & \dots & \frac{\partial f_N}{\partial x_N}(x) \end{pmatrix}$$

and by $J_f(x)$ the determinant of $f'(x)$. We denote by $\mathfrak{R}$ the class of all triples $(f, \Omega, b)$ such that $f \in C(\overline{\Omega}, \mathbb{R}^N) \bigcap C^1(\Omega, \mathbb{R}^N)$ with $\Omega$ a bounded open set in $\mathbb{R}^N$, $b \notin f(\partial\Omega)$ and where $b$ is a regular value of $f$ in $\Omega$, namely $f'(x)$ is invertible for every $x \in \Omega$ satisfying $f(x) = b$.

The definition of the Brouwer degree is given through several steps.

*Step 1.* Definition of the Brouwer degree in the class $\mathfrak{R}$. Assume that $(f, \Omega, b) \in \mathfrak{R}$. The set $\{u \in \Omega : f(u) = b\}$ is finite because $b$ is a regular value of $f$ and $\Omega$ is bounded. In this case one defines the degree by setting

$$\deg(f, \Omega, b) := \sum_{f(x)=b} \operatorname{sign}(J_f(x)). \tag{4.1}$$

*Example 4.1.1* Consider a linear, invertible[1] continuous map $L : \mathbb{R}^N \to \mathbb{R}^N$. Let $\lambda_j$ $(j = 1, \dots, k)$ denote the characteristic values of $L$ (see Definition 1.3.4). If $I$ denotes the identity map in $\mathbb{R}^N$, we claim that

$$\deg(I - L, B_r, 0) = (-1)^{\beta}, \quad r > 0, \tag{4.2}$$

where $\beta$ is the sum of the *algebraic multiplicities* $m_j$ of $\lambda_j \in (0, 1)$.

First let us remark that 0 is a regular value of $f = I - L$ because $\lambda = 1$ is not a characteristic value of $L$. For the same reason the only solution of $f(x) = 0$ is $x = 0$ and (4.1) becomes

$$\deg(I - L, B_r, 0) = \operatorname{sign}(J_{I-L}(0)). \tag{4.3}$$

[1] Or, equivalently, that $\lambda = 1$ is not a characteristic value of $L$.

In order to evaluate $\mathrm{sign}\,(J_{I-L}(0))$, let us begin by pointing out that the eigenvalues $\alpha_j$ of $I - L$ are given by

$$\alpha_j = \frac{\lambda_j - 1}{\lambda_j}.$$

Moreover, the algebraic multiplicity of $\alpha_j$ is equal to $m_j$, namely to that of $\lambda_j$. If we write $I - L$ in its Jordan normal form, then its determinant is given by

$$J_{I-L}(0) = \prod_j \alpha_j = \prod_j \frac{\lambda_j - 1}{\lambda_j}. \tag{4.4}$$

Here it is understood that each $\alpha_j$ (or $\lambda_j$) is repeated $m_j$ times. Now, remark that none of $\alpha_j$ is zero because $\lambda_j \neq 1$ by assumption. In addition, the only $\alpha_j$ which contribute to the $\mathrm{sign}\,[J_{I-L}(0)]$ are the *real* $\alpha_j < 0$, namely the $\lambda_j \in (0, 1)$. This is clear if $\alpha_j > 0$. Moreover, if one $\alpha_j$ is complex, its complex conjugate $\alpha_j^*$ is also an eigenvalue of $I - L$. Hence their product is positive and does not change the sign of $J_{I-L}(0)$. Therefore, (4.3) yields

$$\deg\,(I - L, B_r, 0) = \mathrm{sign}\,[J_{I-L}(0)] = (-1)^\beta\,, \quad \text{where } \beta = \sum_{0<\lambda_j<1} m_j.$$

This immediately implies (4.2). □

*Step 2.* Extension to continuous maps. To extend the definition of degree to singular values $b$ of $f$, one uses the Sard lemma.

**Sard Lemma** *Let $f \in C^1(\Omega, \mathbb{R}^N)$ and consider the set $\mathfrak{S}(f)$ of singular points of $f$, i.e., $\mathfrak{S}(f) = \{x \in \Omega \,:\, J_f(x) = 0\}$. Then the set of singular values, $f(\mathfrak{S}(f))$, has zero Lebesgue measure.*

As a direct consequence, the class of functions $f \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$ for which $b$ is a regular value is dense in the space $C(\overline{\Omega}, \mathbb{R}^N)$, and therefore the degree defined for $\mathfrak{R}$ is uniquely extended to a continuous map in the class of triples $(f, \Omega, b)$ with $f \in C(\overline{\Omega}, \mathbb{R}^N)$, $\Omega$ a bounded open set in $\mathbb{R}^N$ and $b \notin f(\partial\Omega)$.

The main properties of the Brouwer degree are the following.

(P1) *Normalization property*: $\deg\,(I, \Omega, b) = 1$, for $b \in \Omega$, where $I$ is the identity map.

(P2) *Additivity property*: If $\Omega_1$ and $\Omega_2$ are open, bounded disjoint subsets in $\Omega$ and $b \notin f(\overline{\Omega} \setminus (\Omega_1 \cap \Omega_2))$, then $\deg\,(f, \Omega, b) = \deg\,(f, \Omega_1, b) + \deg\,(f, \Omega_2, b)$.

(P3) *Homotopy property*: Let $H \in C([0, 1] \times \overline{\Omega}, \mathbb{R}^N)$ be a homotopy. If $b \in C([0, 1], \mathbb{R}^N)$ satisfies $b(t) \notin H(t, \partial\Omega)$, for every $t \in [0, 1]$, then $\deg\,(H(t, .), \Omega, b)$ is constant. In particular, $\deg\,(H(0, .), \Omega, b(0)) = \deg\,(H(1, .), \Omega, b(1))$.

Actually, it has been proved in [5] that the degree is uniquely determined by the additivity, homotopy and normalization properties. In addition, we list below other properties of the Brouwer degree that can be deduced from (P1)–(P3). We prove the first three properties and leave as an exercise to the reader the remaining ones.

(P4) *Solution property*: If $\deg(f, \Omega, b) \neq 0$, then $b \in f(\Omega)$, namely there exists $x \in \Omega$ such that $f(x) = b$.

*Proof* If $b \notin f(\Omega)$, applying (P2) with $\Omega_1 = \Omega_2 = \emptyset$, we have $\deg(f, \Omega, b) = 2\deg(f, \emptyset, b) = 0$, where the latter equality is also a consequence of (P1) with $\Omega_1 = \Omega$ and $\Omega_2 = \emptyset$. □

(P5) *Excision property*: Let $K \subset \Omega$ be any compact set such that $b \notin f(K)$. Then $\deg(f, \Omega, b) = \deg(f, \Omega \setminus K, b)$.

*Proof* Apply (P2) with $\Omega_1 = \Omega \setminus K$ and $\Omega_2 = \emptyset$. □

(P6) *Dependence on the boundary values*: $\deg(f, \Omega, b)$ depends only on the values of $f$ on $\partial\Omega$, namely if $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and $g \in C(\overline{\Omega}, \mathbb{R}^N)$ are such that $f_{|\partial\Omega} = g_{|\partial\Omega}$, then $\deg(f, \Omega, b) = \deg(g, \Omega, b)$.

*Proof* Apply (P3) with $H(t, x) = tf(x) + (1 - t)g(x)$ and $b(t) = b$. □

(P7) If $\Omega \subset \mathbb{R}^N$ and $f \in C(\overline{\Omega}, \mathbb{R}^n)$, with $N \geq n$, then $\deg(f, \Omega, b) = \deg(f|_{\overline{\Omega}\cap\mathbb{R}^n}, \Omega \cap \mathbb{R}^n, b)$. Here we identify $\mathbb{R}^n$ with the subset $\mathbb{R}^n \times \{0\} \times \overset{(N-n)}{\cdots} \times \{0\}$ of $\mathbb{R}^N$.

(P8) *Continuity with respect to b*: The degree is constant for $b$ on each connected component of $\mathbb{R}^N - f(\partial\Omega)$.

(P9) *Continuity with respect to f*: There exists a neighborhood $U$ of $f$ in $C(\Omega, \mathbb{R}^N)$ such that

$$\deg(f, \Omega, b) = \deg(g, \Omega, b) \quad \forall g \in U.$$

Let us remark that the neighborhood $U$ can be chosen in such a way that $b \notin g(\partial\Omega)$ for any $g \in U$. Hence the $\deg(g, \Omega, b)$ is well defined.

## 4.2 The Leray–Schauder Topological Degree

The Brouwer degree has been extended by Leray and Schauder to spaces of infinite dimension for compact perturbations of the identity.

Let $X$ be a Banach space, and let $\Omega$ be an open subset of $X$. Consider a *compact map* $T \in C(\overline{\Omega}, X)$ and let $\Phi = I - T$ where $I$ denotes the identity in $X$. Let $b \in \Omega$ be such that $b \notin \Phi(\partial\Omega)$. The Leray–Schauder topological degree (for short, LS degree) is defined on any triple $(\Phi, \Omega, b)$ with the above properties. The class of such $(\Phi, \Omega, b)$ is denoted by $\mathfrak{D}$.

*Remark 4.2.1* If $X = \mathbb{R}^N$ we recover the Brouwer degree defined in the previous section. As a consequence, the results (and the proofs) given below can be translated to results for the Brouwer degree.

Since $T$ is compact, the set $\Phi(\partial\Omega)$ is closed and thus

$$\delta := \text{dist}\,(b, \Phi(\partial\Omega)) > 0. \tag{4.5}$$

Moreover, still using the fact that $T$ is compact, we infer that there exists a sequence of continuous operators $T_n$ with finite-dimensional range, namely $T_n(\overline{\Omega}) \subset \mathbb{R}^n$, such that $T_n \to T$, uniformly, (cf. [36, Sect. 6.1]).

Setting $\Phi_n = I - T_n$ and using (4.5) we deduce that

$$\text{dist}\,(b, \Phi_n(\partial\Omega)) > \delta/2 > 0,$$

provided $n$ is sufficiently large and hence $\deg(\,(I - T_n)|_{\Omega\cap\mathbb{R}^n}, \Omega \cap \mathbb{R}^n, b)$ makes sense.

Moreover, using property (P7) of the Brouwer degree, there exists $n_0 \geq 1$ such that

$$\deg((I - T_n)|_{\Omega\cap\mathbb{R}^n}, \Omega \cap \mathbb{R}^n, b) = \text{const.} \quad \forall\, n \geq n_0. \tag{4.6}$$

In addition, this constant is independent of the considered approximation $T_n$ of $T$, as is shown in the following result.

**Lemma 4.2.2** *If $S_n \in C(\overline{\Omega}, \mathbb{R}^n)$ is such that $S_n \to T$ uniformly, then*

$$\deg(\,(I - S_n)|_{\Omega\cap\mathbb{R}^n}, \Omega \cap \mathbb{R}^n, b) = \deg(\,(I - T_n)|_{\Omega\cap\mathbb{R}^n}, \Omega \cap \mathbb{R}^n, b), \quad \forall\, n \gg 1.$$

*Proof* Since both $S_n$ and $T_n$ converge to $T$, then $\|S_n - T_n\| \to 0$ as $n \to \infty$. Using property (P9) of the Brouwer degree the lemma follows. □

According to these remarks, the following definition is in order.

**Definition 4.2.3** If $(\Phi, \Omega, b) \in \mathfrak{D}$, then we set

$$\deg(\Phi, \Omega, b) = \lim_{n\to\infty} \deg((I - T_n)|_{\Omega\cap\mathbb{R}^n}, \Omega \cap \mathbb{R}^n, b).$$

All the properties of the Brouwer degree hold for the LS degree provided that $(\Phi, \Omega, b) \in \mathfrak{D}$. The reader can check this claim as an exercise. Without changing notation, we will still refer to these properties as (P1)–(P9).

It is convenient to state explicitly the form that takes the homotopy property.

**Proposition 4.2.4** (Homotopy property) *Let $T \in C([0,1] \times \overline{\Omega}, X)$ be such that $T(t,.)$ is a compact map for all $t \in [0,1]$. Define $\Phi_t(u) = u - T(t,u)$. If $b : [0,1] \longrightarrow X$ is continuous and $\Phi_t(u) \neq b(t)$ for every $t \in [0,1]$ and $u \in \partial\Omega$, then*

$$\deg(\Phi_t, \Omega, b(t)) = \textit{const.} \quad \forall t \in [0,1]. \qquad □$$

*Example* An interesting homotopy which we will consider is $T(t,u) = t\,T(u)$ with $T$ compact and $b(t) \equiv b \notin \Phi_t(\partial\Omega)$ for every $t \in [0,1]$. In this case,

$$\deg(I - T, \Omega, b) = \deg(I, \Omega, b) = 1.$$

A more general version of the homotopy property is stated below without proof.

**Proposition 4.2.5.** (General homotopy property) *Let $\Omega$ be a bounded, open subset of $\mathbb{R} \times X$ and let $T : \overline{\Omega} \to X$ be a compact map. For every $t \in \mathbb{R}$ we consider the $t$-slice*

$$\Omega_t = \{u \in X : (t,u) \in \Omega\}$$

*and the map $\Phi_t : \overline{\Omega_t} \to X$ given by*

$$\Phi_t(u) = u - T(t,u).$$

*If*

$$\Phi_t(u) \neq b, \quad \forall u \in \partial\Omega_t,$$

*then the topological degree* $\deg(\Phi_t, \Omega_t, b)$ *is well defined and independent of $t$.* □

An application of the homotopy property of the LS degree is the fixed point theorem proved by Juliusz Schauder [80].

**Theorem 4.2.6** (Schauder fixed point theorem) *If $B$ is a closed ball of a real Banach space $X$ and $T : B \longrightarrow B$ is compact, then $T$ has a fixed point.*

*Proof* Without loss of generality we can assume that $B$ is the closed ball $\overline{B}_r$ of center 0 and radius $r$. Observe that the thesis of the theorem is clearly verified if $0 \in (I - T)(\partial B)$. On the other hand, if for each $u \in \partial B$, $Tu \neq u$, then, using in addition that

$$t\|Tu\| < r = \|u\|, \quad \forall t \in [0,1), \quad \forall u \in \partial B,$$

we deduce for $\Phi_t = I - tT$ that $0 \notin \Phi_t(\partial B)$, for every $t \in [0,1]$. By applying the homotopy property of the degree to the family of compact operators $\Phi_t$ we get

$$\deg(\Phi_1, B, 0) = \deg(I, B, 0) = 1.$$

By the existence property, this implies that $\Phi_1$ has a zero, i.e., $T$ has a fixed point in $B$. □

*Remark 4.2.7* The key issue in the above proof has been to establish that

$$\{u : u - tT(u) = 0 \text{ for some } t \in [0,1]\} \subset B_r \tag{4.7}$$

to apply the homotopy property. The condition (4.7) means that $r$ is an *a priori bound* of the solutions of the equation $u - tT(u) = 0$.

*Remark 4.2.8* As a consequence of the Dugundji extension theorem, every closed convex set $D$ of a normed linear space $X$ is a retract (i.e., there exists a map $R : X \longrightarrow X$ such that $Rx = x$ for every $x \in D$), and the Schauder theorem is also true if we substitute the closed ball $B$ by any closed bounded convex set $D$. Indeed, it suffices to consider a closed ball $B$ containing $D$ and to apply the above theorem to the composition operator $T \circ R : B \longrightarrow D \subset B$.

### 4.2.1 Index of an Isolated Zero and Computation by Linearization

Assume that, for $\Phi \in C(\overline{\Omega}, \mathbb{R}^N)$, $u_0 \in \Omega$ is an isolated solution of the equation $\Phi(u) = 0$, i.e., a unique solution of this equation in a neighborhood (say $B_{r_0}(u_0) = \{u \in \mathbb{R}^N : \|u - u_0\| < r_0\} \subset \Omega$) of $u_0$. We deduce then that $(\Phi, B_r(u_0), 0) \in \mathfrak{D}$ and from the excision property that

$$\deg(\Phi, B_r(u_0), 0) = \deg(\Phi, B_{r_0}(u_0), 0), \quad \forall r \in (0, r_0).$$

This allows us to define the index of $\Phi$ relative to $u_0$ by setting

$$\mathrm{i}(\Phi, u_0) = \lim_{r\to 0} \deg(\Phi, B_r(u_0), 0).$$

Below, if $\Phi = I - T$ is $C^1$, we show how to evaluate the index of $\Phi$ relative to $u_0$ through the index of its derivative at zero. Up to a translation, we can assume, without loss of generality, that $u_0 = 0$. As it has been mentioned, in the sequel it is always understood that the triples considered are $(\Phi, B_\varepsilon, 0) \in \mathfrak{D}$.

We begin with some preliminary remarks. First of all it is well known that if $T$ is compact, then the linear map $T'(0)$ is also compact (see, e.g., [15, Lemma 3.17]). If, in addition, $\Phi'(0) = I - T'(0)$ is invertible ($\lambda = 1$ is not a characteristic value of $T'(0)$), then zero is the unique zero of $\Phi'(0)$ and, consequently, $\deg(\Phi'(0), B_\varepsilon, 0)$ is well defined.

**Lemma 4.2.9** *Suppose that* $\Phi$ *is of class* $C^1$ *such that* $\det[\Phi'(0)] \neq 0$. *Then*

$$\deg(\Phi, B_\varepsilon, 0) = \deg(\Phi'(0), B_\varepsilon, 0),$$

*for every sufficiently small* $\varepsilon > 0$.

*Remark 4.2.10* Under the hypothesis of the previous lemma, there exists a small $\varepsilon > 0$ such that the equation $\Phi(u) = 0$ has a unique solution $u = 0$ in the ball $B_\varepsilon$ centered at $u = 0$ with radius $\varepsilon$.

*Proof* Consider the family of maps

$$H(t,u) = \begin{cases} \dfrac{1}{t}\Phi(tu), & \text{if } t \in (0,1], \\ \\ \Phi'(0)u, & \text{if } t = 0. \end{cases}$$

Clearly $H(t,u)$ is an admissible continuous homotopy. Otherwise, there exists $(t^*, u^*) \in [0,1] \times \partial\Omega$ such that $H(t^*, u^*) = 0$. From the definition of $H$ and since $0 \notin \partial\Omega$ it follows that $t^* \in (0,1)$. Hence we have $\Phi(t^* u^*) = 0$, a contradiction. Then the result follows by applying the homotopy property. □

Let $\chi(0, 1, T'(0))$ denote the set of all characteristic values $\lambda \in (0,1)$ of $T'(0)$. Since $T'(0)$ is compact, the set $\chi(0,1,T'(0))$ is finite (see Theorem 1.3.6). Moreover, let $mult(\lambda)$ be the algebraic multiplicity of $\lambda$.

**Theorem 4.2.11** *Let $T$ be of class $C^1(\overline{\Omega}, X)$ and compact. Moreover we suppose that $T'(0)$ is invertible. Then there holds*

$$\mathrm{i}\,(\Phi, 0) = \mathrm{i}\,(\Phi'(0), 0) = (-1)^{\beta},$$

*where*[2]

$$\beta = \sum_{\lambda \in \chi(0,1,T'(0))} \mathrm{mult}\,(\lambda).$$

*Proof* Let $V \subset X$ be the space spanned by the eigenfunctions corresponding to the $\lambda$'s in $\chi(0, 1, T'(0))$. Then $V$ has dimension $\beta$ and there exists $W \subset X$ such that $X = V \oplus W$. Let $P, Q$ be the projections onto $V, W$, respectively.

We claim that the homotopy $H(t, u) = (1 - t)(u - T'(0)u) + t(-Pu + Qu)$ (which is a linear map of the type *Identity—Compact* since $-P + Q = I - 2P$ where the range of $P$ is finite dimensional) is admissible on $B_\epsilon$ (actually, on any ball $B_r$). Indeed, arguing by contradiction, suppose there exists $(t^*, u^*) \in [0, 1] \times \partial B_\epsilon$ such that $H(t^*, u^*) = 0$. Writing $v = Pu^* \in V$ and $w = Qu^* \in W$ and using that $V$ and $W$ are invariant by $T'(0)$, this means that

$$(1 - 2t^*)v = (1 - t^*)T'(0)v$$

$$w = (1 - t^*)T'(0)w.$$

Observe that $(1 - 2t^*)(1 - t^*)^{-1} < 1 < (1 - t^*)^{-1}$ for $t \in (0, 1)$ and thus, since $T'(0)|_V$ has only eigenvalues greater than one, $v = 0$. Similarly, since $T'(0)|_W$ does not have eigenvalues greater than one, $w = 0$; i.e., $u^* = 0$, a contradiction. As a consequence, by the homotopy invariance, Definition 4.2.3 and (4.3), we obtain

$$\deg\,(I - T'(0), B_\varepsilon, 0) = \deg\,(-P + Q, B_\varepsilon, 0) = \deg\,(I - 2P, B_\varepsilon, 0) = (-1)^{\beta}.$$

□

## 4.3 Continuation Theorem of Leray–Schauder

### *4.3.1 A Topological Lemma*

The following separation lemma (see [44, 87]) will be useful.

**Lemma 4.3.1** *Let $(M, d)$ be a compact metric space, let $A$ be a connected component of $M$ and let $B$ be a closed subset of $M$ such that $A \cap B = \emptyset$. Then there exist compact sets $M_A$ and $M_B$ satisfying*

- *$A \subset M_A$, $B \subset M_B$.*
- *$M = M_A \cup M_B$ and $M_A \cap M_B = \emptyset$.*

---

[2] If $\chi(0, 1, T'(0)) = \emptyset$ we set $\beta = 0$.

The proof of Lemma 4.3.1 is based on the notion of $\varepsilon$-chainable points.

**Definition 4.3.2** Given $\varepsilon > 0$, we say that two points $a, b$ in a compact metric space $M$ are $\varepsilon$-chainable if there exists a finite number of points $x_1, \ldots, x_n \in M$ such that $x_1 = a,\ x_n = b$ and $d(x_1, x_{i+1}) < \varepsilon$ for every $i = 1, ..., n-1$.

Clearly the relation "to be $\varepsilon$-chainable" is an equivalence relation. Using this it is easily proved that the set $A_\varepsilon(a)$ of all points $x$ in $M$ which are $\varepsilon$-chainable is open and closed in $M$. As a consequence, if $A$ is a connected set in $M$, then $A \subset A_\varepsilon(a)$ for every $a \in A$ and, by the transitivity of the relation, the set $A_\varepsilon(a)$ does not depend on the choice of the point $a \in A$. We denote this set as $A_\varepsilon$ in this case and we prove the following result.

**Proposition 4.3.3** *Let $A$ be a connected set in $M$ and let $\varepsilon > 0$. If $A_\varepsilon$ denotes the set of points in $M$ which are $\varepsilon$-chainable with some point in $A$ (thus, with all points in $A$), then $M_0 := \bigcap_{\varepsilon > 0} A_\varepsilon$ is connected.*

*Proof* We begin by proving that every two points in $M_0$ are $\delta$-chainable for every $\delta > 0$. Indeed, let $b_1, b_2 \in M_0$ and $\delta > 0$. By the definition of $M_0$, each one of the points $b_i$ $(i = 1, 2)$ is $\delta$-chainable with every point in $A$. The transitive property implies then that $b_1$ and $b_2$ are $\delta$-chainable.

Now, assume, by contradiction, that $M_0$ is not connected, i.e., that there exist closed sets $C_1, C_2$ in $M_0$ such that

$$M_0 = C_1 \cup C_2, \quad C_1 \cap C_2 = \emptyset.$$

Since $M_0$ is closed (by intersection of closed sets) in the compact $M$, we deduce that $C_1$ and $C_2$ are also disjoint compact. Let $\delta = \operatorname{dist}(C_1, C_2) > 0$. Clearly, every two points $c_1 \in C_1$ and $c_2 \in C_2$ are not $\delta$-chainable, contradicting our first assertion at the beginning of the proof. Therefore, $M_0$ is connected and the proof is concluded. □

Now we are ready to prove the separation Lemma 4.3.1.

*Proof of Lemma 4.3.1* It suffices to show that there exists $\varepsilon > 0$ such that

$$B \cap A_\varepsilon = \emptyset. \tag{4.8}$$

Indeed, if the existence of this $\varepsilon$ has been stated, then we can take $M_A = A_\varepsilon$, $M_B = M \setminus A_\varepsilon$.

To prove (4.8), we argue by contradiction: assume that $B \cap A_\varepsilon \neq \emptyset$, for every $\varepsilon > 0$. This means that fixing $a \in A$ and taking, for $n \in \mathbb{N}$, $\varepsilon_n = \frac{1}{n}$, there exists $b_n \in B$ such that $a$ and $b_n$ are $\varepsilon_n$-chainable.

By the compactness of $B$, there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ which converges to some $b \in B$. We claim that $b \in A_{\varepsilon_k}$, for every $k \in \mathbb{N}$. Indeed, given $k \in \mathbb{N}$, we can choose $n_k > k$ such that $d(b_{n_k}, b) < \varepsilon_k = \frac{1}{k}$ and hence $b$ is $\varepsilon_k$-chainable with $b_{n_k}$. Using also that $n_k > k$, we have $b_{n_k} \in A_{\varepsilon_{n_k}} \subset A_{\varepsilon_k}$ and $b_{n_k}$ is $\varepsilon_k$-chainable with every point in $A$. The claim is proved then by applying the transitive property. Consequently, $b \in \bigcap_{k\in\mathbb{N}} A_{\varepsilon_k} = M_0$, where the set $M_0$ is connected by Proposition 4.3.3 and contains $A$. Taking into account that $A$ is a connected component, we get $M_0 = A$ and therefore $b_0 \in M_0 = A$ which implies that $A \cap B \neq \emptyset$, a contradiction proving (4.8) and the lemma. □

### *4.3.2 A Theorem by Leray and Schauder*

Let $X$ be a real Banach space, let $\Omega$ be a bounded and open subset of $X$, let $a < b$ and let $T : [a, b] \times \overline{\Omega} \to X$ be a compact map. For $\lambda \in [a, b]$, consider the equation

$$\Phi(\lambda, u) = u - T(\lambda, u) = 0, \quad u \in X. \tag{4.9$_\lambda$}$$

Observe that $T$ can be seen as a family of compact operators

$$T_\lambda(u) := T(\lambda, u), \quad u \in X.$$

Similarly, we denote $\Phi_\lambda = I - T_\lambda$. Define

$$\Sigma = \{(\lambda, u) \in [a, b] \times \overline{\Omega} : \Phi(\lambda, u) = 0\}.$$

We use the notation $\Sigma_\lambda$ for the $\lambda$-slice of $\Sigma$, i.e., $\Sigma_\lambda = \{u \in \overline{\Omega} : (\lambda, u) \in \Sigma\}$.

**Theorem 4.3.4** (Leray–Schauder [64], (see also [50])) *Assume that $X$ is a real Banach space, $\Omega$ is a bounded, open subset of $X$ and $\Phi : [a, b] \times \overline{\Omega} \longrightarrow X$ is given by $\Phi(\lambda, u) = u - T(\lambda, u)$ with $T$ a compact map. Suppose also that*

$$\Phi(\lambda, u) = u - T(\lambda, u) \neq 0, \quad \forall (\lambda, u) \in [a, b] \times \partial\Omega.$$

*If*

$$\deg(\Phi_a, \Omega, 0) \neq 0, \tag{4.10}$$

*then*

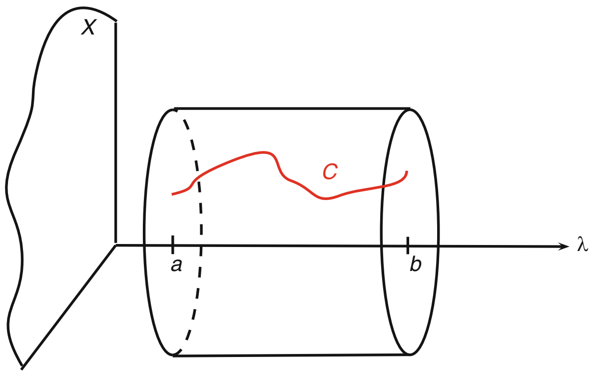
1. $(4.9)_\lambda$ *has a solution in $\Omega$ for every $a \leq \lambda \leq b$.*
2. *Furthermore, there exists a compact connected set $\mathcal{C} \subset \Sigma$ such that*

$$\mathcal{C} \cap (\{a\} \times \Sigma_a) \neq \emptyset \text{ and } \mathcal{C} \cap (\{b\} \times \Sigma_b) \neq \emptyset, \text{ (see Fig. 4.1).}$$

*Proof* 1. First, observe that the homotopy property of the degree implies that

$$\deg(\Phi_\lambda, \Omega, 0) = \text{ const.}, \quad \forall \lambda \in [a, b].$$

**Fig. 4.1** Leray–Schauder theorem




Therefore, by (4.10), the constant is not zero. Thus, if $\lambda \in [a, b]$ then $\deg(\Phi_\lambda, \Omega, 0) \neq 0$ and, in particular, from the existence property, $(4.9)_\lambda$ has a solution $u_\lambda$.

2. We argue by contradiction, supposing that every connected component set $\mathcal{C} \subset \Sigma$ containing points of $\{a\} \times \Sigma_a$ does not intersect $\{b\} \times \Sigma_b$, (see Fig. 4.2). Applying Lemma 4.3.1 we deduce that there exist two disjoint compact sets $M_a \supset \mathcal{C} \supset \{a\} \times \Sigma_a$ and $M_b \supset \{b\} \times \Sigma_b$ such that $\Sigma = M_a \cup M_b$. It follows that there exists a bounded open set $\mathcal{O}$ in $[a, b] \times X$ such that $\{a\} \times \Sigma_a \subset \mathcal{C} \subset M_a \subset \mathcal{O}$, $M_b \cap \mathcal{O} = \emptyset$ and $T(\lambda, u) \neq u$ for $u \in \partial\mathcal{O}_\lambda$, with $\lambda \in [a, b]$. (We are denoting by $\mathcal{O}_\lambda$ the $\lambda$-slice of $\mathcal{O}$, i.e., $\mathcal{O}_\lambda = \{u : (\lambda, u) \in \mathcal{O}\}$.)

The general homotopy property of the degree implies that

$$\deg(\Phi_\lambda, \mathcal{O}_\lambda, 0) = \deg(\Phi_a, \mathcal{O}_a, 0)$$

for $a \leq \lambda \leq b$. By (4.10) we deduce that $\deg(\Phi_b, \mathcal{O}_b, 0) = 0$ for every $\lambda \in [a, b]$. However, since $\Phi_b$ has no zeros in $\mathcal{O}_b$, we get a contradiction, proving case 2. □

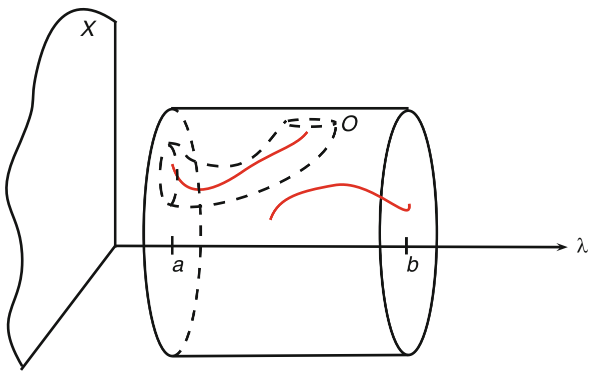



**Fig. 4.2** Proof of Leray–Schauder theorem by contradiction

## 4.4 Other Continuation Theorems

Let $X$ be a real Banach space and consider a compact map $T : \mathbb{R} \times X \to X$. We denote again by $\Sigma$ the closed set of the pairs $(\lambda, u) \in \mathbb{R} \times X$ with $u$ a solution of $(4.9)_\lambda$. We prove the existence of continua of solutions in $\Sigma$.

**Theorem 4.4.1** *For $\lambda_0 \in \mathbb{R}$, let $u_0 \in X$ be an isolated solution of the problem* $(4.9)_{\lambda_0}$ *such that*

$$\mathrm{i}(\Phi_{\lambda_0}, u_0) \neq 0.$$

*Then the connected component of $\Sigma$ that contains $(\lambda_0, u_0)$ is not bounded in $\mathbb{R} \times X$.*

*Proof* We argue by contradiction and assume that the connected component, $C$, of $\Sigma$ that contains $(\lambda_0, u_0)$ is bounded. Since $u_0$ is isolated, there exists $\delta_1 > 0$ such that

$$(\{\lambda_0\} \times B_{\delta_1}(u_0)) \cap \Sigma = \{(\lambda_0, u_0)\}. \tag{4.11}$$

For $0 < \delta < \delta_1$, let $U_\delta$ be a $\delta$-neighborhood of $C$, that is

$$U_\delta = \{(\lambda, u) \in \mathbb{R} \times X : \operatorname{dist}((\lambda, u), C) < \delta\}.$$

As in the proof of the previous theorem, we can take $\mathcal{O} \subset \mathbb{R} \times X$ with $\partial\mathcal{O} \cap \Sigma = \emptyset$, $(\lambda_0, u_0) \in \mathcal{O}$. Indeed, in the case $\Sigma \cap \partial U_\delta = \emptyset$, it suffices to choose $\mathcal{O} = U_\delta$. In the other case, since the set $K = \overline{U}_\delta \cap \Sigma$ is a compact metric space, we can apply Lemma 4.3.1 to the closed sets $C$ and $\Sigma \cap \partial U_\delta$ to deduce the existence of two disjoint compact sets $A, B$ of $K$ such that

$$K = A \cup B,\ C \subset A.$$

Taking a neighborhood of $A$ as $\mathcal{O}$ we conclude the claim. Hence, the topological degree $\deg(I - T_\lambda, \mathcal{O}_\lambda, 0)$ is well defined. Further, using the general homotopy property we derive that $\deg(I - T_\lambda, \mathcal{O}_\lambda, 0)$ is constant for values of $\lambda$ in a compact interval. On the other hand, since $\mathcal{O}$ is bounded in $\mathbb{R} \times X$, there exists $\varepsilon_1 \in \mathbb{R}^+$ such that

$$\mathcal{O}_\lambda = \emptyset \quad \text{if } \lambda \notin (\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1),$$

which, using that the degree relative to the empty set is zero, implies that

$$\deg(I - T_\lambda, \mathcal{O}_\lambda, 0) = 0,\ \forall \lambda \in \mathbb{R},$$

and hence that $\deg(I - T_{\lambda_0}, \mathcal{O}_{\lambda_0}, 0) = 0$. But, by the excision property of the degree and (4.11), $0 = \deg(I - T_{\lambda_0}, \theta_{\lambda_0}, 0) = \mathrm{i}(\Phi_{\lambda_0}, u_0)$, contradicting the hypothesis $\mathrm{i}(\Phi_{\lambda_0}, u_0) \neq 0$. □

The next result is useful to prove existence of a continuum with a specific shape (see Fig. 4.3).

**Theorem 4.4.2** *Let $U \subset X$ be bounded, open and let $a, b \in \mathbb{R}$ be such that* $(4.9)_\lambda$ *has no solution in $\partial U$, for every $\lambda \in [a, b]$, and that* $(4.9)_b$ *has no solution in $\overline{U}$. Let $U_1 \subset$*

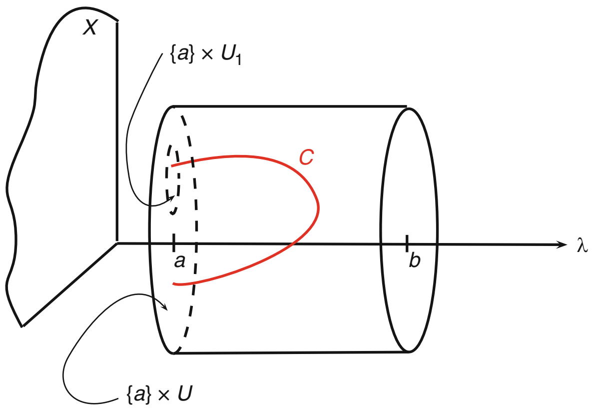



**Fig. 4.3** Theorem 4.4.2

*$U$ be open such that* $(4.9)_a$ *has no solution in* $\partial U_1$ *and* $\deg(I - T_a, U_1, 0) \neq 0$. *Then there exists a continuum $C$ in* $\Sigma = \{(\lambda, u) \in [a, b] \times X : u$ *is a solution of* $(4.9)_\lambda\}$, *such that*

$$C \cap (\{a\} \times U_1) \neq \emptyset, \quad C \cap (\{a\} \times (U \setminus U_1)) \neq \emptyset.$$

*Proof* We use the following notation:

$$K = ([a, b] \times U) \cap \Sigma,$$

$$A = (\{a\} \times \overline{U}_1) \cap K,$$

$$B = (\{a\} \times \overline{(U \setminus U_1)}) \cap K.$$

Since $(4.9)_b$ has no solution in $\overline{U}$ and $K$ is compact, we can consider $K \subset [a, s] \times \overline{U}$ for some $s \in (a, b)$.

We argue by contradiction and assume that the theorem is false. By Lemma 4.3.1, there exist disjoint, compact subsets $K_A, K_B$ containing respectively $A$ and $B$, such that $K = K_A \cup K_B$. Let $\mathcal{O}$ be a $\delta$-neighborhood of $K_A$ such that $\text{dist}(\mathcal{O}, K_B) > 0$. Hence the Leray–Schauder degree is well defined in $\mathcal{O}_\lambda = \{u \in \overline{U} : (\lambda, u) \in \mathcal{O}\}$ for every $\lambda \in [a, b]$. Furthermore, by the general homotopy property, we have

$$\deg(I - T_\lambda, \mathcal{O}_\lambda, 0) = \text{constant},$$

and consequently

$$\deg(I - T_a, \mathcal{O}_a, 0) = \deg(I - T_b, \mathcal{O}_b, 0). \tag{4.12}$$

On the other hand, since $\mathcal{O} \cap K_B = \emptyset$, there are no solutions of Eq. $(4.9)_a$ in $\mathcal{O}_a \setminus \overline{U}_1$ and hence, by the excision property, we deduce that

$$\deg(I - T_a, \mathcal{O}_a, 0) = \deg(I - T_a, U_1, 0) \neq 0.$$

However, by hypothesis we know that $\mathcal{O}_b = \emptyset$, and thus we conclude that $\deg(I - T_b, O_b, 0) = 0$. This is a contradiction with (4.12), proving the theorem. □

# Chapter 5
# An Outline of Critical Points

This chapter deals with variational methods. In addition to the existence of minima of a functional, we discuss the mountain pass theorem, and the linking theorem which are used to find saddle points. A perturbation method, variational in nature, is studied in the last section.

## 5.1 Definitions

Let $E$ be a Hilbert space and $\mathcal{J} \in C^1(E, \mathbb{R})$. Then the Fréchet derivative $d\mathcal{J}(u)$ is a linear continuous map from $E$ to $\mathbb{R}$ and hence we can define, by the Riesz theorem, the *gradient* $\mathcal{J}'(u) \in E$ of $\mathcal{J}$ at $u$ by setting

$$(\mathcal{J}'(u) \mid v) = d\mathcal{J}(u)[v], \quad \forall\, v \in E.$$

*Example 5.1.1* (i) If $E = \mathbb{R}^N$ and $F \in C^1(\mathbb{R}^N, \mathbb{R})$, the gradient $F'(x)$ is nothing but the vector in $\mathbb{R}^N$ with components $F_{x_i}(x)$, $i = 1, \dots, N$.
(ii) If $E$ is a Hilbert space with norm $\|\cdot\|$ and scalar product $(\cdot \mid \cdot)$, for the functional $\mathcal{J}(u) = \frac{1}{2}\|u\|^2$ one has $d\mathcal{J}(u)[v] = (u \mid v)$ and hence $\mathcal{J}'(u) = u$.
(iii) More in general, if $A$ is a linear symmetric operator on $E$ and $\mathcal{J}(u) = \frac{1}{2}(Au \mid u)$, one has $d\mathcal{J}(u)[v] = (Au \mid v)$ and hence $\mathcal{J}'(u) = Au$.

An operator $T : E \to E$ is called *variational* if there exists a differentiable functional $\mathcal{J} : E \to \mathbb{R}$ such that $T = \mathcal{J}'$.

A *critical point* of $\mathcal{J}$ is a $u \in E$ such that $\mathcal{J}'(u) = 0$. A critical value $c$ is a number $c \in \mathbb{R}$ for which there exists a critical point $u \in E$ with level $\mathcal{J}(u) = c$. We will see that, in our applications, critical points are (weak) solutions of differential equations. Therefore, if $T$ is a variational operator, in order to find the solutions of $T(u) = 0$ it suffices to look for the critical points of $\mathcal{J}$, where $\mathcal{J}' = T$.

Below, we will limit ourselves to consider two classical results dealing with the existence of *minima* and of *saddle points* of the *mountain pass type*.

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82, DOI 10.1007/978-0-8176-8114-2_5, © Springer Science+Business Media, LLC 2011

## 5.2 Minima

Usually the existence of minima of a functional is deduced as a consequence of the Weierstrass theorem (see Theorem 1.2.4). For instance, by Corollary 1.2.5, *every functional* $\mathcal{J} \in C^1(E, \mathbb{R})$ *which is coercive* ($\lim_{\|u\| \to \infty} \mathcal{J}(u) = +\infty$) *and weakly lower semicontinuous (w.l.s.c.)* ($\mathcal{J}(u) \leq \liminf \mathcal{J}(u_n)$ *if* $u_n \rightharpoonup u$) *is bounded from below and has a global minimum*, so that we have the following example.

*Example 5.2.1* Consider the functional

$$\mathcal{J}(u) = \tfrac{1}{2}\|u\|^2 - \mathcal{H}(u),$$

where $\mathcal{H} \in C^1(E, \mathbb{R})$ is weakly continuous (namely $u_n \rightharpoonup u \Rightarrow \mathcal{H}(u_n) \to \mathcal{H}(u)$) and satisfies

$$|\mathcal{H}(u)| \leq a_1 + a_2\|u\|^\alpha,$$

with $a_1, a_2 > 0$ and $\alpha < 2$. Then

$$\mathcal{J}(u) \geq \tfrac{1}{2}\|u\|^2 - a_1 - a_2\|u\|^\alpha$$

and since $\alpha < 2$, it follows that $\mathcal{J}$ is coercive. It is well known that the norm $\|u\|$ is w.l.s.c. This and the fact that $\mathcal{H}$ is weakly continuous implies that $\mathcal{J}$ is w.l.s.c. Therefore Corollary 1.2.5 applies and yields a global minimum $z \in E$ of $\mathcal{J}$ such that $\mathcal{J}'(z) = 0$, i.e., $z = \mathcal{H}'(z)$.

Dealing with nonlinear eigenvalue problems, we shall also consider minima constrained on a submanifold $M$ of $E$. We will focus on the specific situation for $M = \mathcal{G}^{-1}(0)$, where $\mathcal{G} \in C^{1,1}(E, \mathbb{R})$. If $\mathcal{G}'(u) \neq 0$ on $M$, then $M$ is a smooth manifold in $E$.

We say that $u \in M$ is a local minimum constrained on $M$ for the functional $\mathcal{J} \in C^1(E, \mathbb{R})$ if there exists a neighborhood $U$ of $u$ such that

$$\mathcal{J}(u) \leq \mathcal{J}(v), \quad \forall\, v \in U \cap M.$$

If $u$ is a local minimum of $\mathcal{J}$ on $M$, then there exists $\lambda \in \mathbb{R}$ such that $\mathcal{J}'(u) = \lambda \mathcal{G}'(u)$. The proof is quite similar to the elementary finite-dimensional case (see Exercise 24). The value $\lambda$ is called the Lagrange multiplier.

*Example 5.2.2* If $\mathcal{G}(u) = \frac{1}{2}(\|u\|^2 - R^2)$, then $M$ is a sphere of radius $R$ and a local minimum of $\mathcal{J}$ on $M$ satisfies $\mathcal{J}'(u) = \lambda u$.

## 5.3 The Mountain Pass Theorem

The mountain pass theorem deals with the existence of critical points of a functional $\mathcal{J} \in C^1(E, \mathbb{R})$ which has a strict local minimum at, say, $u = 0$. Specifically, we assume that it satisfies the following two "geometric" assumptions.

$(J1)$ There exist $r, \rho > 0$ such that $\mathcal{J}(u) \geq \rho$ for all $u \in E$ with $\|u\| = r$.
$(J2)$ $\exists\, v \in E$, $\|v\| > r$, such that $\mathcal{J}(v) \leq 0 = \mathcal{J}(0)$.

*Example 5.3.1* Let $\mathcal{J} \in C^1(E, \mathbb{R})$ be a functional of the form

$$\mathcal{J}(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2}(Au \mid u) - \mathcal{H}(u),$$

where $A$ is a compact linear bounded symmetric operator in $E$ and $\mathcal{H} \in C^2(E, \mathbb{R})$ is homogeneous of degree $\alpha > 2$, namely $\mathcal{H}(tu) = t^\alpha \mathcal{H}(u)$ for all $t \geq 0$ and all $u \in E$. This implies that $\mathcal{H}(0) = 0$ and $\mathcal{H}'(0) = 0$ as well as $d^2\mathcal{H}(0)[v, v] = 0$ for all $v \in E$. Therefore one has that $\mathcal{J}'(0) = 0$ and

$$d^2\mathcal{J}(0)[v, v] = \|v\|^2 - (Av \mid v) - d^2\mathcal{H}(0)[v, v] = \|v\|^2 - (Av \mid v).$$

From this we infer that $d^2\mathcal{J}(0)$ is positive definite provided $(Av \mid v) < \|v\|^2$ for all $v \in E$, i.e., if $\|A\| < 1$ (since $\|A\| = \sup_{\|v\|=1}(Av \mid v)$ because $A$ is symmetric). Consequently, by Theorem 1.3.6, $d^2\mathcal{J}(0)$ is positive definite iff all the eigenvalues of $A$ are smaller than 1. If this holds, $u = 0$ is a strict local minimum for $\mathcal{J}$, i.e., $(J1)$ is satisfied. Moreover, suppose that $\mathcal{H} \not\equiv 0$ and let $v \neq 0$ be such that $\mathcal{H}(v) \neq 0$. The following holds:

$$\mathcal{J}(tv) = \tfrac{1}{2}t^2\,\|v\|^2 - \tfrac{1}{2}t^2\,(Av \mid v) - t^\alpha\,\mathcal{H}(v). \tag{5.1}$$

If $\mathcal{H}(v) > 0$, resp. $\mathcal{H}(v) < 0$, (5.1) and the fact that $\alpha > 2$ implies that there exists $t^* > 0$, resp. $t^* < 0$, such that $\mathcal{J}(t^*v) < 0$ and $(J2)$ holds.

Let $\mathcal{J} \in C^1(E, \mathbb{R})$ be a functional satisfying the assumptions $(J1)$–$(J2)$. Without loss of generality, we can also assume (to simplify notation) that $\mathcal{J}(0) = 0$. Consider the class of all paths joining $u = 0$ and $u = v$,

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0,\ \gamma(1) = v\}$$

and set

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}(\gamma(t)). \tag{5.2}$$

Clearly, the class $\Gamma$ is not empty and, by $(J1)$–$(J2)$, $c \geq \rho > 0$. We expect that there exists at least a critical point of $\mathcal{J}$ at the min–max level $c$. However, even in finite dimension, this is in general false without the assumption of additional hypotheses. Indeed, for $E = \mathbb{R}^2$ we have the following example due to Brezis and Nirenberg. The functional $\mathcal{J}(x, y) = x^2 + (1 - x)^3 y^2$ has a unique critical point, which is the origin $(0, 0)$. Since $\mathcal{J}(x, y) = x^2 + y^2 + o(x^2 + y^2)$ as $(x, y) \to (0, 0)$, it follows that $(0, 0)$ is a strict local minimum for $\mathcal{J}$, namely $(J1)$ holds. Moreover

$$\mathcal{J}(t, t) = t^2 + (1 - t)^3 t^2 \to -\infty, \quad \text{as } t \to +\infty$$

and hence $(J2)$ holds, too.

The additional hypothesis that we need in order to show that $c$ is a critical value is the following "compactness" condition:

$(PS)_c$ Every sequence $\{u_n\}$ such that
$(i)$ $\mathcal{J}(u_n) \to c$,
$(ii)$ $\mathcal{J}'(u_n) \to 0$,

has a converging subsequence.

This condition is usually called the (local) Palais–Smale condition at level $c$ and the sequences $\{u_n\}$ satisfying $(i)$–$(ii)$ are called $(PS)_c$ sequences.

Notice that $(PS)_c$ is equivalent to the following two conditions:

(a) If the sequence $\{u_n\}$ satisfies $\{\mathcal{J}(u_n)\} \longrightarrow c$ and $\|u_n\| \longrightarrow +\infty$, then there exists $c > 0$ such that $\|\mathcal{J}'(u_n)\| \geq c$ for $n$ sufficiently large.
(b) Every bounded sequence $\{u_n\}$ with $\{\mathcal{J}(u_n)\} \longrightarrow c$ and $\{\mathcal{J}'(u_n)\} \longrightarrow 0$ possesses a convergent subsequence.

The second one is a compactness condition which is satisfied in many cases (see Lemma 7.1.1), while the first condition means that every sequence of points $\{u_n\}$ with level near $c$ ($\{\mathcal{J}(u_n)\} \longrightarrow c$) and that are almost critical points ($\{\mathcal{J}'(u_n)\} \longrightarrow 0$) are a priori bounded, i.e., there exists $M > 0$ such that $\|u_n\| \leq M$.

The reader should observe that (a) is more general than the standard a priori estimate for the solutions of the variational equation $\mathcal{J}'(u) = 0$ (see Remark 4.2.7). Actually, there are examples in which a functional $\mathcal{J}$ has an unbounded sequence of critical points but satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$ (see [15, 18, 77]).

Given $a \in \mathbb{R}$, let us consider the sublevel $\mathcal{J}^a = \{u \in E : \mathcal{J}(u) \leq a\}$ of $\mathcal{J}$. The Palais–Smale condition allows us to deform sublevels $\mathcal{J}^a$ of the functional $\mathcal{J}$. Specifically, we have the following deformation lemma.

**Lemma 5.3.2** *Suppose that $b \in \mathbb{R}$ is not a critical value of $\mathcal{J} \in C^{1,1}(E, \mathbb{R})$ and that $(PS)_b$ holds. Then there exist $\delta > 0$ and a map $\eta \in C(E, E)$ such that $\eta(\mathcal{J}^{b+\delta}) \subset \mathcal{J}^{b-\delta}$. Moreover, $\eta(u) = u$ for all $u \in \mathcal{J}^{b-2\delta}$.*

*Proof* The $(PS)_b$ condition and the assumption that $b$ is not a critical value of $\mathcal{J}$ mean that there exists $\delta > 0$ satisfying

$$\|\mathcal{J}'(u)\| \geq \delta, \quad \forall\, u \in \mathcal{J}^{-1}([b - \delta, b + \delta]).$$

Thus, we can construct ([76]) a vector field $W \in C^{0,1}(E, E)$ in such a way that

$$W(u) = \begin{cases} -\mathcal{J}'(u)\|\mathcal{J}'(u)\|^{-2}, & \text{if } b + \delta \geq \mathcal{J}(u) \geq b - \delta, \\ 0, & \text{if } \mathcal{J}(u) \leq b - 2\delta, \end{cases}$$

and consider the Cauchy problem

$$\phi' = W(\phi), \quad \phi(0) = u.$$

Since $W$ is bounded, it is easy to check that the flow $\phi^t(u)$ is defined for all $t \geq 0$. Let us point out that for any $u \in \mathcal{J}^{b+\delta}$ the following holds (the dependence on $u$ is understood):

$$\frac{d\mathcal{J}(\phi^t)}{dt} = (\mathcal{J}'(\phi^t), (\phi^t)') = (\mathcal{J}'(\phi^t), W(\phi^t)) \le 0. \tag{5.3}$$

In particular, $\mathcal{J}(\phi^t(u))$ is decreasing with respect to $t \ge 0$. Take $T = 2\delta$. We claim that $\eta(u) = \phi^T(u)$ is the map we are looking for. Otherwise, for some $u \in \mathcal{J}^{b+\delta} - \mathcal{J}^{b-\delta}$, we have $\mathcal{J}(\phi^T(u)) > b - \delta$ and $W(\phi^s(u)) = -\mathcal{J}'(u)\|\mathcal{J}'(u)\|^{-2}$ for all $0 \le s \le T$. Hence (5.3) yields $(\mathcal{J}'(\phi^s), W(\phi^s)) = -1$, and

$$\mathcal{J}(\phi^T(u)) = \mathcal{J}(\phi^0(u)) - T = \mathcal{J}(u) - T \le b + \delta - T = b - \delta,$$

which is a contradiction, proving the claim. The last statement follows immediately from the fact that $W \equiv 0$ on $\mathcal{J}^{b-2\delta}$. □

*Remark 5.3.3* The preceding proof highlights that

$$\left.\begin{array}{c} b \text{ is not a critical value} \\ \text{and } (PS)_b \text{ holds} \end{array}\right\} \Leftrightarrow \exists\delta > 0 \,:\, \|\mathcal{J}'(u)\| \ge \delta,\ \forall u \in \mathcal{J}^{-1}([b-\delta, b+\delta]).$$

*Remark 5.3.4* Using the notion of *pseudo-gradient vector field*, the hypothesis that $\mathcal{J}$ is of class $C^{1,1}$ can be weakened by assuming that $\mathcal{J}$ is $C^1$. For details we refer to [15, pp. 120–123].

*Remark 5.3.5* A deformation lemma for a functional $\mathcal{J}$ constrained on a smooth manifold $M = \mathcal{G}^{-1}(0) \subset E$ can also be proved. It suffices to substitute $\mathcal{J}'$ with the constrained gradient

$$\nabla_M \mathcal{J}(u) = \mathcal{J}'(u) - \frac{(\mathcal{J}'(u), \mathcal{G}'(u))}{\|\mathcal{G}'(u)\|^2}\mathcal{G}'(u), \tag{5.4}$$

which is nothing but the projection of $\mathcal{J}'(u)$ on the tangent space $T_u M = \{v \in E : (\mathcal{G}'(u) \mid v) = 0\}$.

We are now in position to prove the mountain pass theorem

**Theorem 5.3.6** (Mountain pass) *If $\mathcal{J} \in C^1(E, \mathbb{R})$ satisfies $(J1)$–$(J2)$ and $(PS)_c$ holds, then $c \ge \rho > 0$ is a positive critical value for $\mathcal{J}$. Precisely, there exists $z \in E$ such that $\mathcal{J}(z) = c > 0$ and $\mathcal{J}'(z) = 0$. In particular, $z \ne 0$ and $z \ne v$.*

*Proof* If, by contradiction, there is no critical point at level $c$, then Lemma 5.3.2 and Remark 5.3.4 allow us to find $\varepsilon \in (0, \frac{c}{2})$ and a continuous map $\eta : E \to E$ such that $\mathcal{J}(\eta(u)) \le c - \varepsilon$, for all $u \in E$ such that $\mathcal{J}(u) \le c + \varepsilon$. Moreover, $\eta$ is such that $\eta(u) = u$ provided $\mathcal{J}(u) \le c - 2\varepsilon$. In particular, $\eta(0) = 0$ and $\eta(v) = v$. By the definition of $c$, there exists $\gamma \in \Gamma$ such that $\max_{t\in[0,1]} \mathcal{J}(\gamma(t)) < c + \varepsilon$. As a consequence, the path $\eta \circ \gamma$ belongs to $\Gamma$. On the other hand, $\max_{t\in[0,1]} \mathcal{J}(\eta \circ \gamma(t)) \le c - \varepsilon < c$, which contradicts the definition of $c$. □

*Remark 5.3.7* (i) $\mathcal{J}$ can be unbounded from above and from below.

(ii) The mountain pass critical point is, in general, a saddle point: if it is non-degenerate, then its Morse index is 1. By definition, a critical point $u$ of $\mathcal{J}$ is non-degenerate if $\mathcal{J}''(u)$ is invertible. In such a case, its Morse index is, by definition, the number of eigenvalues of $\mathcal{J}''(u)$ smaller than 0.
(iii) Observe that the proof is based on the fact that every continuous curve in $E$ joining $u = 0$ with $v$ has to cross the sphere $\|u\| = r$.

Expanding the preceding remark (iii), we say that a closed subset $S$ of $E$ *links* the relative boundary $\partial Q$ of a submanifold $Q$ of $E$ if $S \cap \partial Q = \emptyset$ and for every map $h \in C(Q, E)$, such that $h(u) = u$ for every $u \in \partial Q$, there holds $h(Q) \cap S \neq \emptyset$.

There are other situations in which it is possible to find a min–max critical value of a functional $\mathcal{J}$ with properties similar to the mountain pass. Let us focus on the following case, which will be used in the sequel to find solutions of an elliptic equation. Fixing $\overline{u} \in E \setminus \{0\}$, suppose that

$$\lim_{|t|\to\infty} \mathcal{J}(t\overline{u}) = -\infty. \tag{5.5}$$

Let $W$ denote the subspace orthogonal to $\mathbb{R}\overline{u}$ and assume that

$$\inf\{\mathcal{J}(w) : w \in W\} > -\infty. \tag{5.6}$$

Consider the class of paths

$$\widetilde{\Gamma} = \{\gamma \in C([0,1], E) : \gamma(0) = -t\overline{u},\ \gamma(1) = t\overline{u}\},$$

where $t \gg 1$ is taken in such a way that $\mathcal{J}(\pm t\overline{u}) < \inf\{\mathcal{J}(w) : w \in W\}$ and define

$$\widetilde{c} := \inf_{\gamma\in\widetilde{\Gamma}} \max_{s\in[0,1]} \mathcal{J}(\gamma(s)),$$

**Theorem 5.3.8** *If* (5.5) *and* (5.6) *hold and the Palais–Smale condition holds at the level* $\widetilde{c}$*, then* $\widetilde{c}$ *is a critical value of* $\mathcal{J}$*.*

*Proof* Each $\gamma \in \widetilde{\Gamma}$ crosses $W$ and hence $\widetilde{c}$ is a finite number greater than $\inf\{\mathcal{J}(w) : w \in W\}$. Repeating the arguments carried out to prove the mountain pass theorem, it follows that $\widetilde{c}$ is a critical value for $\mathcal{J}$ provided the Palais–Smale condition holds at the level $\widetilde{c}$. □

Observe that the hypotheses (5.5) and (5.6) imply that $W$ and a large sphere of $\mathbb{R}\overline{u}$ link. More generally, it was proved by P.H. Rabinowitz that the result is also true if the sphere is taken in a finite-dimensional subspace.

**Theorem 5.3.9** (Saddle point theorem) *Let* $E = V \oplus W$ *with* $V$ *finite dimensional. Assume also that* $\mathcal{J} \in C^1(E, \mathbb{R})$ *satisfies for some* $R > 0$ *that*

$$\rho := \inf\{\mathcal{J}(w) : w \in W\} > \max_{v\in V,\, \|v\|=R} \mathcal{J}(v),$$

*and* $(PS)_{\widetilde{c}}$ *holds. If* $\Gamma$ *is the set of all continuous maps* $h$ *from the ball in* $V$ *of radius* $R$ *and center* 0 *into* $E$ *such that its restriction to the boundary of the ball is the identity, then* $\widetilde{c} = \inf_{h\in\Gamma} \max_{\|v\|=R} \mathcal{J}(h(v)) \geq \rho$ *is a critical value for* $\mathcal{J}$*.*

*Proof* It suffices to observe that $W$ and the sphere in $V$ of radius $R$ are linked (see Exercise 20) and to use similar arguments to the previous ones. The details are left to the reader. □

## 5.4 The Ekeland Variational Principle

As another application of the deformation Lemma 5.3.2 we will prove a version of the Ekeland variational principle [46]. For simplicity, we will consider a specific case dealing with a Hilbert space and a $C^1$ functional.

**Theorem 5.4.1** *Let $E$ be a Hilbert space and let $\mathcal{J} \in C^1(E, \mathbb{R})$ be bounded from below. Then:*

1. *For every $\delta > 0$, there exists $u \in E$ such that $\mathcal{J}(u) \leq \inf_E \mathcal{J} + \delta$ and $\|\mathcal{J}'(u)\| \leq \delta$.*
2. *In particular, if $(PS)_c$ holds for the level $c = \inf_E \mathcal{J}$, then $\mathcal{J}$ attains its infimum.*

*Proof* Set $b := \inf_E \mathcal{J}$. If the assertion in case 1 does not hold, then an application of Lemma 5.3.2 and Remark 5.3.3 allows the construction of $\eta \in C(E, E)$ which maps the sublevel $\mathcal{J}^{b+\delta}$ into $\mathcal{J}^{b-\delta}$, contradicting the definition of $b$. With respect to the proof of case 2, it is sufficient to observe that by choosing $\delta_n = 1/n$, $n \in \mathbb{N}$, we find a minimizing Palais–Smale sequence at level $\inf_E \mathcal{J}$. □

The complete assertion of the general Ekeland principle [46] is the following one.

**Theorem 5.4.2** *Assume that $(X, d)$ is a complete metric space and that $\mathcal{J} : X \to \mathbb{R} \cup \{+\infty\}$ is a l.s.c. functional bounded from below with $\mathcal{J} \not\equiv +\infty$. If, for some $\varepsilon > 0$, a point $u_\varepsilon \in X$ satisfies $\mathcal{J}(u_\varepsilon) < \inf_X \mathcal{J} + \varepsilon$, then there exists $v_\varepsilon \in X$ such that*

$$\mathcal{J}(v_\varepsilon) \leq \mathcal{J}(u_\varepsilon),$$

$$d(u_\varepsilon, v_\varepsilon) \leq 1,$$

$$\mathcal{J}(z) > \mathcal{J}(v_\varepsilon) - \varepsilon d(v_\varepsilon, z), \quad \forall z \neq v_\varepsilon.$$

□

*Remark 5.4.3* 1. Notice that in general the functional $\mathcal{J}$ does not have to attain its infimum. However, the above theorem states that the pertubed functional $\mathcal{J}(z) + \varepsilon d(v_\varepsilon, z)$ does attains its infimum (at $v_\varepsilon$).
2. It is also possible to use the general Ekeland principle to give an alternative proof of the mountain pass Theorem 5.3.6. Indeed, we will follow this approach in Chap. 12 to extend this theorem to functionals which are differentiable along some particular directions.

Similar arguments to those used in Theorem 5.4.1 can be applied to prove the existence of minima constrained on a submanifold $M = \mathcal{G}^{-1}(0) \subset E$ such that $\mathcal{G} \in C^{1,1}(E, \mathbb{R})$. In this case, we say that the functional $\mathcal{J}$ constrained on $M$ satisfies the Palais–Smale condition $(PS)_c$ at level $c$ if every sequence $\{u_n\}$ in $M$ such that $\mathcal{J}(u_n) \to c$ and $\nabla_M \mathcal{J}(u_n) \to 0$ has a converging subsequence.

**Theorem 5.4.4** *Let $\mathcal{J} \in C^1(E, \mathbb{R})$ be bounded from below on $M = \mathcal{G}^{-1}(0)$ where $\mathcal{G}'(u) \neq 0$ on $M$. Let $m := \inf_M \mathcal{J}(u) > -\infty$ and suppose that $(PS)_m$ holds. Then any minimizing sequence has a converging subsequence. In particular, there exists $u \in M$ and $\lambda \in \mathbb{R}$ such that $\mathcal{J}(u) = m$ and $\mathcal{J}'(u) = \lambda \mathcal{G}'(u)$.*

*Proof* Let $w_n$ be a minimizing sequence for $\mathcal{J}$ constrained on $M$. Then by Lemma 5.3.2 and Remark 5.3.5, there exists $u_n \in M$ such that $\|u_n - w_n\| \to 0$, $\mathcal{J}(u_n) \to m$ and $\nabla_M \mathcal{J}'(u_n) \to 0$. Using the $(PS)_m$ condition it follows that $u_n$ (and so $w_n$) converges (up to a subsequence) to some $u \in M$. Obviously $\mathcal{J}(u) = m$ and $\nabla_M \mathcal{J}(u) = 0$, i.e., by (5.4), $\mathcal{J}'(u) = \lambda \mathcal{G}'(u)$ with $\lambda = (\mathcal{J}'(u), \mathcal{G}'(u))/\|\mathcal{G}'(u)\|^2$. □

## 5.5 Another Min–Max Theorem

The mountain pass theorem can be extended to cover the case in which $u = 0$ is not a local minimum but a saddle point. As before, we assume without loss of generality that $\mathcal{J}(0) = 0$.

Let $E = V \oplus W$, where $V$ is a closed subspace with $\dim(V) = k < +\infty$ and $W = V^{\perp}$. We denote by $S_W(r)$ the sphere in $W$ of radius $r$, i.e., $S_W(r) = \{w \in W : \|w\| = r\}$. We consider the following hypotheses (see Fig. 5.1).

$(J3)$ There exist $r, \rho > 0$ such that

$$\mathcal{J}(w) \geq \rho, \quad \forall w \in S_W(r).$$

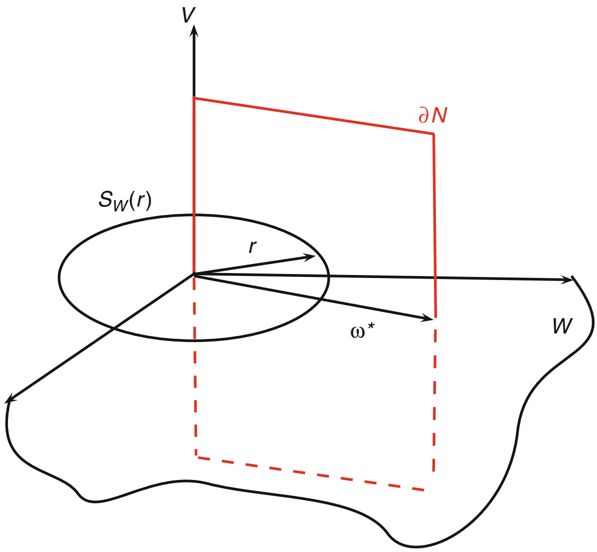



**Fig. 5.1** Linking hypotheses $(J3)$ and $(J4)$

$(J4)$ There exist $R > 0$ and $w^* \in W$, with $\|w^*\| > r$ such that, letting $N = \{u = v + sw^* : v \in V, \|v\| \le R, s \in [0,1]\}$, one has that

$$\mathcal{J}(u) \le 0, \quad \forall u \in \partial N.$$

*Example 5.5.1* Completing Example 5.3.1, consider again the functional $\mathcal{J}$ given by $\mathcal{J}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}(Au \mid u) - \mathcal{H}(u)$, with $\mathcal{H}$ $\alpha$-homogeneous for some $\alpha > 2$. Let $\mu_1 > \mu_2 \ge \mu_3 \ge \cdots$ denote the characteristic values of $A$. If $e_i \neq 0$ is such that $\mu_i Ae_i = e_i$, it follows that $(\mathcal{J}''(0)e_i \mid e_i) = \|e_i\|^2 - (Ae_i \mid e_i) = (1 - \mu_i^{-1})\|e_i\|^2$. Hence, letting $V = \text{span}\{e_1, \ldots, e_k\}$, we deduce that $\mathcal{J}''(0)$ is positive definite on $W = V^\perp$, provided $\mu_{k+1} < 1$, and this suffices to find $r, \rho > 0$ such that

$$\mathcal{J}(w) \ge \rho, \quad \forall w \in S_W(r),$$

and thus $(J3)$ holds.

On the other hand, assuming in addition that $1 \le \mu_k$ and $\mathcal{H} \ge 0$, we can also see that $(J4)$ holds as well. Indeed, let us fix $\widehat{w} \in W$ with $\|\widehat{w}\| = 1$ and set $\widehat{V} = V \oplus \mathbb{R}\widehat{w}$. Then, for all $\widehat{v} \in \widehat{S} = \{\widehat{v} \in \widehat{V} : \|\widehat{v}\| = 1\}$, one has

$$\mathcal{J}(t\widehat{v}) = \left[\frac{1}{2} - \frac{1}{2}(A\widehat{v} \mid \widehat{v})\right] t^2 - t^\alpha \mathcal{H}(\widehat{v}), \quad \forall t \ge 0.$$

Since $\widehat{V}$ is finite dimensional, there exists $\widehat{t} > 0$, depending only on the dimension $k+1$ of $\widehat{V}$, such that $\mathcal{J}(t\widehat{v}) \le 0$ for all $t \ge \widehat{t}$ and all $\widehat{v} \in \widehat{S}$. Let $R = \max\{r, \widehat{t}\}$. Take $w^* = R\widehat{w}$ and consider the set $N$ defined in $(J4)$. The preceding argument shows that $\mathcal{J}(u) \le 0$ for all $u = v + sw^*$ on the part of $\partial N$ with $s > 0$. Moreover, $\mathcal{J}$ is also smaller than or equal to zero on the part of the boundary of $N$ with $s = 0$. Actually, since $1 \le \mu_k$, $\mathcal{J}''(0) = I - A$ is semi-negative defined on $V$, and then $\mathcal{J}(v) = \frac{1}{2}[\|v\|^2 - (Av \mid v)] - \mathcal{H}(v) \le 0$. This proves that $(J4)$ holds.

The set $N$ can be identified with $\{u = (v,s) \in V \times [0,1] : \|v\| \le R\}$. Extending the class $\Gamma$, we consider the class of maps

$$\Gamma_k = \{g \in C(N, E) : g(v,s) = (v,s), \quad \forall (v,s) \in \partial N\}.$$

**Lemma 5.5.2** *For any $g \in \Gamma_k$, there exists $u_g \in N$ such that $g(u_g) \in S_W(r)$.*

*Proof* If $g \in \Gamma_k$ we set $g(v,s) = (v', s')$. Define the auxiliary map $g^* \in C(N, V \times \mathbb{R})$ by setting

$$g^*(v,s) = (v', s'\|w^*\| - r).$$

By the definition of $\Gamma_k$, we have that $g(v,s) = (v,s)$ for all $(v,s) \in \partial N$. Then

$$g^*(v,s) = (v, s\|w^*\| - r), \quad \forall (v,s) \in \partial N. \tag{5.7}$$

This and $\|w^*\| > r$ imply that

$$g^*(v,s) \neq (0,0), \quad \forall (v,s) \in \partial N. \tag{5.8}$$

Actually, if the first component of $g^*(v, s)$ is zero for some $(v, s) \in \partial N$, then (5.7) yields $v = 0$. Hence $s = 1$ and, still using (5.7), we find that the second component of $g^*(v, s)$ is $\|w^*\| - r > 0$. From (5.8) it follows that we can evaluate the topological degree $\deg(h^*, N_0, (0, 0))$, where $N_0 = \mathrm{int}(N)$ is the interior of $N$. Consider the map $\widehat{g}(v, s) = (v, s\|w^*\| - r)$ and note that $\deg(\widehat{g}, N_0, (0, 0)) = 1$, because $\|w^*\| > r$. Since $g(v, s) = \widehat{g}(v, s)$ on $\partial N$, one infers from property (P6) of the degree (see Chap. 4) that

$$\deg(g^*, N_0, (0, 0)) = \deg(\widehat{g}, N_0, (0, 0)) = 1,$$

and there exists $(v_g, s_g) \in N$ such that $g^*(v_g, s_g) = (0, 0)$. By the definition of $g^*$ this means that $u_g = (v_g, s_g)$ verifies: $g(u_g) \in W$ and $\|g(u_g)\| = r$. □

**Theorem 5.5.3** *Let $\mathcal{J} \in C^1(E, \mathbb{R})$ satisfy $(J3)$–$(J4)$ and, setting*

$$c = \inf_{g \in \Gamma_k} \max_{u \in N} \mathcal{J}(g(u)),$$

*suppose that $(PS)_c$ holds. Then there exists $z \in E$ such that $\mathcal{J}(z) = c > 0$ and $\mathcal{J}'(z) = 0$.*

*Proof* By Lemma 5.5.2, for any $g \in \Gamma_k$, there exists $u_g \in N$ such that $g(u_g) \in S_W(r)$. Therefore, by $(J3)$,

$$\max_{u \in N} \mathcal{J}(g(u)) \geq \mathcal{J}(g(u_g)) \geq \rho > 0, \quad \forall\, g \in \Gamma_k,$$

and this implies that $c \geq \rho > 0$. The rest of the proof is similar to that of the mountain pass theorem. □

*Remark 5.5.4* If $V = \{0\}$, then Theorem 5.5.3 becomes the mountain pass theorem. Actually, if $V = \{0\}$, the class $\Gamma_k$ is nothing but $\Gamma$ and $c$ coincides with the mountain pass critical value.

Theorem 5.5.3 is a specific case of more general results which are referred to as *linking theorems*. A linking theorem in which $V$ has infinite dimension has been proven in [30].

## 5.6 Some Perturbation Results

Let $E$ be a Hilbert space, $\mathcal{I} \in C^2(E, \mathbb{R})$ and let $\mathcal{G} \in C^2(\mathbb{R} \times E, \mathbb{R})$ be a family of functionals depending on a real parameter $\varepsilon$. We are interested in the critical points of

$$\mathcal{I}_\varepsilon(u) = \mathcal{I}(u) + \mathcal{G}_\varepsilon(u),$$

where $\mathcal{G}_\varepsilon(u) = \mathcal{G}(\varepsilon, u)$. The functional $\mathcal{I}$ plays the role of the *unperturbed functional* and $\mathcal{G}$ is the *perturbation*. The specific situation we are interested in is the case in which the unperturbed functional has a finite-dimensional manifold $Z$ of critical points and we look for these $z \in Z$ from which emanate solutions of $\mathcal{I}'_\varepsilon = 0$.

The fact that $\mathcal{G}$ plays the role of a perturbation term is formulated in the following assumption:

(A1) $\qquad \|\mathcal{G}_\varepsilon'(z)\| \to 0, \qquad \text{as} \quad \varepsilon \to 0, \quad \text{uniformly in } z \in Z.$

Let $T_z Z$ denote the tangent space to $Z$ at $z$, let $W = (T_z Z)^\perp$ and let $P$ denote the projection from $E$ onto $W$. Writing $u = z + w$, with $z \in Z$ and $w \in W$, and using a Lyapunov–Schmidt reduction, the equation $\mathcal{I}_\varepsilon'(u) = 0$ is equivalent to the system

$$\begin{cases} P\mathcal{I}_\varepsilon'(z+w) = 0, \\ (I - P)\mathcal{I}_\varepsilon'(z+w) = 0. \end{cases} \tag{5.9}$$

The former equations are nothing but forms of the *auxiliary equation*.

We further assume that there exists $c > 0$ such that, for $\varepsilon$ small enough,

(A2) $\qquad \|[P\mathcal{I}_\varepsilon''(z)]^{-1}\| \le c, \quad \text{uniformly in } z \in Z.$

For any $z \in Z$ fixed, assumption (A2) allows us to define the map $S_\varepsilon : B_{\varepsilon,c} \longrightarrow W$ by setting

$$S_\varepsilon(w) = w - [P\mathcal{I}_\varepsilon''(z)]^{-1}(P\mathcal{I}_\varepsilon'(z+w)), \tag{5.10}$$

where $B_{\varepsilon,c}$ denotes the ball

$$B_{\varepsilon,c} = \{w \in W : \ \|w\| \le 2c\|\mathcal{G}_\varepsilon'(z)\|\}.$$

*Remark 5.6.1* (A1) implies that the ball $B_{\varepsilon,c}$ shrinks to $w$ as $\varepsilon \to 0$.

Let us point out that if $w$ is such that $S_\varepsilon(w) = w$ then $u = z + w$ is a solution of the auxiliary equation.

In order to find a fixed point of $S_\varepsilon$ a last assumption is in order:

(A3) $\qquad \|\mathcal{I}_\varepsilon''(z+w) - \mathcal{I}_\varepsilon''(z)\| \le \dfrac{1}{2c}, \quad \text{uniformly in } z \in Z \text{ and } w \in B_{\varepsilon,c}.$

*Remark 5.6.2* (i) If $\mathcal{I}_\varepsilon(u) = \mathcal{I}(u) + \varepsilon\mathcal{G}(u)$, assumptions (A1) and (A2) are trivially verified (see also Remark 5.6.1). As for (A2), it can be substituted by requiring that $P\mathcal{I}''(z)$ be invertible. As we will see in the sequel, the invertibility of $P\mathcal{I}''(z)$ is closely related to a suitable non-degeneracy of the manifold $Z$.
(ii) If $\mathcal{G}(\varepsilon, u) = \varepsilon\mathcal{G}(u)$, the auxiliary equation becomes

$$P\mathcal{I}'(z+w) + \varepsilon\mathcal{G}'(z+w) = 0$$

and can be solved near $Z$ directly by means of the implicit function theorem, provided $P\mathcal{I}''(z)$ is invertible. On the other hand, in some applications, like the one discussed in Sect. 13.2, we need to work with perturbations in the general form $\mathcal{G}(\varepsilon, u)$ for which the implicit function theorem cannot be applied.

Let us show that $S_\varepsilon$ has a fixed point in $B_{\varepsilon,c}$.

**Lemma 5.6.3** *For $\varepsilon$ small enough, $S_\varepsilon(B_{\varepsilon,c}) \subset B_{\varepsilon,c}$ and $S_\varepsilon$ is a contraction. Therefore, $S_\varepsilon$ has a unique fixed point $w_{\varepsilon,z} \in B_{\varepsilon,c}$. In particular, $\|w_{\varepsilon,z}\| \le 2c\|\mathcal{I}'_\varepsilon(z)\|$ and thus $\|w_{\varepsilon,z}\| \to 0$ as $\varepsilon \to 0$.*

*Proof* For $v, w \in B_{\varepsilon,c}$ there exists $\widehat{v}$ belonging to the segment joining $v$ and $w$ such that

$$S_\varepsilon(v) - S_\varepsilon(w) = S'_\varepsilon(\widehat{v})[v-w],$$

for some $\widehat{v} \in B_{\varepsilon,c}$. One has

$$\begin{aligned} S'_\varepsilon(\widehat{v})[v-w] &= v - w - [P\mathcal{I}''_\varepsilon(z)]^{-1}(P\mathcal{I}''_\varepsilon(z+\widehat{v})[v-w]) \\ &= [P\mathcal{I}''_\varepsilon(z)]^{-1}(P\mathcal{I}''_\varepsilon(z)[v-w] - P\mathcal{I}''_\varepsilon(z+\widehat{v})[v-w]). \end{aligned}$$

Using (A3) we infer

$$\begin{aligned} \|S'_\varepsilon(\widehat{v})[v-w]\| &\le \|[P\mathcal{I}''_\varepsilon(z)]^{-1}\| \cdot \|P\mathcal{I}''_\varepsilon(z)[v-w] - P\mathcal{I}''_\varepsilon(z+\widehat{v})[v-w]\| \\ &\le \tfrac{1}{2}\|v-w\| \end{aligned} \tag{5.11}$$

and this suffices to show that $S_\varepsilon$ is a contraction on $B_{\varepsilon,c}$. Furthermore, the following holds:

$$\|S_\varepsilon(0)\| = \|[P\mathcal{I}''_\varepsilon(z)]^{-1}\| \cdot \|P\mathcal{I}'_\varepsilon(z)\| \le c\|\mathcal{G}'_\varepsilon(z)\|.$$

Using this equation and (5.11) it follows that

$$\|S_\varepsilon(w)\| \le \|S_\varepsilon(0)\| + \|S_\varepsilon(w) - S_\varepsilon(0)\| \le c\|\mathcal{G}'_\varepsilon(z)\| + \tfrac{1}{2}\|w\|.$$

For $w \in B_{\varepsilon,c}$ one has that $\|w\| \le 2c\|\mathcal{G}'_\varepsilon(z)\|$ and hence we deduce

$$\|S_\varepsilon(w)\| \le 2c\|\mathcal{G}'_\varepsilon(z)\|.$$

This shows that $S_\varepsilon(B_{\varepsilon,c}) \subset B_{\varepsilon,c}$ and completes the proof. □

It is now convenient to restrict ourselves to a manifold $Z$ with coordinates $\xi \in \mathbb{R}^n$. To simplify the exposition, we will carry out the proof in the case that $\xi = s \in \mathbb{R}$. The general case requires minor changes.

We will write $z_\varepsilon(s)$ to denote the solution of the auxiliary equation found in Lemma 5.6.3 and define the function $w : Z \to W$ by setting $w_\varepsilon(s) = w_{\varepsilon, z_\varepsilon(s)}$. Some properties of $w_\varepsilon(s)$ are stated in the following lemma.

**Lemma 5.6.4** *The function $w_\varepsilon(s)$ is of class $C^1$ and the derivative $w'_\varepsilon(s)$ satisfies $|w'_\varepsilon(s)| \to 0$ as $\varepsilon \to 0$, uniformly with respect to $s \in Z$.*

*Proof* Since $w_\varepsilon(s)$ has been found by using the Banach contraction principle, it easily follows that $w$ is $C^1$. To compute the derivative $w'_\varepsilon$ let us remark that $w_\varepsilon$ satisfies $P\mathcal{I}'_\varepsilon(z + w_\varepsilon) = 0$, namely (we understand the dependence upon $s$)

$$\mathcal{I}'_\varepsilon(z + w_\varepsilon) = \mathcal{I}'_\varepsilon(z + w_\varepsilon)(z' + w'_\varepsilon)\frac{z'}{|z'|^2}.$$

Taking the derivative we get

$$\begin{aligned}\mathcal{I}''_\varepsilon(z+w_\nu e)(z'+w'_\nu e) &= \mathcal{I}''_\varepsilon(z+w_\varepsilon)(z'+w'_\varepsilon)\frac{z'}{|z'|^2}\\ &\quad + \mathcal{I}'_\varepsilon(z+w_\varepsilon)(z'+w'_\varepsilon)z''\frac{z'}{|z'|^2}\\ &\quad + \mathcal{I}'_\varepsilon(z+w_\varepsilon)(z'+w'_\nu e)z'\frac{d}{ds}\left(\frac{z'}{|z'|^2}\right).\end{aligned}$$

Let us evaluate separately each term on the right-hand side. Since $w'_\varepsilon$ is orthogonal to $z'$ and $w_\varepsilon \to 0$, we find

$$\mathcal{I}''_\varepsilon(z+w_\varepsilon)(z'+w'_\varepsilon)\frac{z'}{|z'|^2} = \mathcal{I}''_\varepsilon(z+w_\varepsilon) = \mathcal{I}''_\varepsilon(z) + O(\varepsilon).$$

Furthermore, using (A1) and again the fact that $w_\varepsilon \to 0$, we infer

$$\mathcal{I}'_\varepsilon(z+w_\varepsilon) = \mathcal{I}'_\varepsilon(z) + \mathcal{I}'_\varepsilon(z+\theta w_\varepsilon) = O(\varepsilon), \quad (\theta \in [0,1]).$$

From the preceding estimates we deduce that

$$P\mathcal{I}''_\varepsilon(z+w_\varepsilon)w'_\varepsilon = o(1).$$

Finally, (A2) yields that $w'_\varepsilon \to 0$ as $\varepsilon \to 0$. □

Using Lemma 5.6.3 we can define the *reduced functional* by setting

$$\widetilde{\mathcal{I}}_\varepsilon(z) = \mathcal{I}_\varepsilon(z + w_{\varepsilon,z}), \quad z \in Z.$$

**Theorem 5.6.4** *Suppose that* (A1)–(A3) *hold. If $z_\varepsilon \in Z$ is a critical point of $\widetilde{\mathcal{I}}_\varepsilon$, then $u_\varepsilon := z_\varepsilon + w_{\varepsilon,z_\varepsilon}$ is a critical point of $\mathcal{I}_\varepsilon$, provided $\varepsilon$ is small enough.*

*Proof* As before we will consider again the case in which $\xi = s \in \mathbb{R}$. With this notation, the reduced functional becomes

$$\widetilde{\mathcal{I}}_\varepsilon(s) = \mathcal{I}_\varepsilon(z(s) + w_{\varepsilon,s}), \quad s \in \mathbb{R},$$

and $s_\varepsilon$ is a critical point of $\widetilde{\mathcal{I}}_\varepsilon$, provided

$$\widetilde{\mathcal{I}}'_\varepsilon(s_\varepsilon) = \mathcal{I}'_\varepsilon(z(s_\varepsilon) + w_{\varepsilon,s_\varepsilon}) \cdot (z'(s_\varepsilon) + w'_{\varepsilon,s_\varepsilon}) = 0. \tag{5.12}$$

Using the auxiliary equation $P\mathcal{I}'_\varepsilon(z(s_\varepsilon) + w_{\varepsilon,s_\varepsilon}) = 0$ we infer that

$$\mathcal{I}'_\varepsilon(z(s_\varepsilon) + w_{\varepsilon,s_\varepsilon}) = a_\varepsilon z'(s_\varepsilon)$$

where

$$a_\varepsilon = \mathcal{I}'_\varepsilon(z(s_\varepsilon) + w_{\varepsilon,s_\varepsilon})z'(s_\varepsilon).$$

Therefore (5.12) becomes

$$a_\varepsilon z'^2(s_\varepsilon) + a_\varepsilon z'(s_\varepsilon)w'_{\varepsilon,s_\varepsilon} = 0.$$

Since $w'_{\varepsilon,s_\varepsilon} \to 0$ as $\varepsilon \to 0$, it follows that, for $\varepsilon$ small enough, $a_\varepsilon = 0$, namely $\mathcal{I}'_\varepsilon(z(s_\varepsilon) + w_{\varepsilon,s_\varepsilon}) = 0$. □

*Remark 5.6.6* If $\mathcal{I}_\varepsilon(u) = \mathcal{I}(u) + \varepsilon\mathcal{G}(u)$ one finds that

$$\begin{aligned}\widetilde{\mathcal{I}}_\varepsilon(s) &= \mathcal{I}(z(s) + w_{\varepsilon,s}) + \varepsilon\mathcal{G}(z + w_{\varepsilon,s})\\ &= \mathcal{I}(z(s)) + (\mathcal{I}'(z(s)), w_{\varepsilon,s}) + \varepsilon\mathcal{G}(z(s)) + \varepsilon(\mathcal{G}'(z(s)), w_{\varepsilon,s}) + o(\|w_{\varepsilon,s}\|).\end{aligned}$$

Since $\mathcal{I}'(z) = 0$ we can use Lemma 5.6.3 to infer

$$\widetilde{\mathcal{I}}_\varepsilon(s) = \mathcal{I}(z(s)) + \varepsilon\mathcal{G}(z(s)) + o(\varepsilon).$$

Since $\mathcal{I}(z) = \text{const.}$, to find the stationary points of $\widetilde{\mathcal{I}}_\varepsilon$ it suffices to look for the stable stationary points, e.g. maxima or minima, of $\mathcal{G}(z(s))$.

Applications of the previous theorem will be given in Chap. 13 to find semiclassical states of nonlinear Schrödinger (NLS) equations with potentials and to prove the existence of standing waves for some nonautonomous systems of coupled NLS equations.

# Chapter 6
# Bifurcation Theory

In this chapter we are concerned with bifurcation theory. We discuss the local bifurcation from a simple eigenvalue found by analytical methods, the bifurcation from an odd eigenvalue by using the topological degree and the Krasnoselskii result on variational operators. The Rabinowitz global bifurcation theorem is also proved.

## 6.1 Local Results

Let $X$ and $Y$ be Banach spaces and consider the equation

$$F(\lambda, u) = 0, \quad u \in X,$$

where $F : \mathbb{R} \times X \longrightarrow Y$ satisfies $F(\lambda, 0) \equiv 0$. We say that $\lambda^*$ is a *bifurcation point* of $F(\lambda, u) = 0$ if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times X$, with $u_n \neq 0$, such that $\lambda_n \longrightarrow \lambda^*$ and $F(\lambda_n, u_n) = 0$.

A particular case is that when $X = Y = E$ is a Hilbert space and the equation is given by

$$Lu + H(u) = \lambda u, \quad u \in E, \tag{6.1}$$

where $L : E \longrightarrow E$ is linear and compact and $H \in C^1(E, E)$ is such that $H(0) = 0$, $H'(0) = 0$.

Denoting by $\Sigma_0$ the set of nontrivial solutions of (6.1), namely

$$\Sigma_0 = \{(\lambda, u) \in \mathbb{R} \times X : Lu + H(u) = \lambda u,\ u \neq 0\},$$

and taking the closure $\Sigma$ of $\Sigma_0$, we see that $\lambda^* \in \mathbb{R}$ is a bifurcation point of (6.1) if and only if $(\lambda^*, 0) \in \Sigma$.

**Lemma 6.1.1** *If $\lambda^*$ is a bifurcation point of* (6.1) *then $\lambda^*$ belongs to the spectrum of $L$.*

*Proof* The equation $F(\lambda, u) := Lu + H(u) - \lambda u = 0$ has the trivial solution $u = 0$ for all $\lambda \in \mathbb{R}$. Since $H'(0) = 0$, one has that $d_u F(\lambda, 0)[v] = Lv + H'(0)[v] - \lambda v =$

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82, DOI 10.1007/978-0-8176-8114-2_6, 

$Lv - \lambda v$. If $\lambda \notin \sigma(L)$, then $d_u F(\lambda, 0)$ is invertible and the implicit function theorem yields a neighborhood of $(\lambda, 0)$ such that the unique solution of (6.1) is $u = 0$. This proves that $\lambda$ is not a bifurcation point. □

We will give below conditions under which an *eigenvalue* of $L$ is a bifurcation point.

### 6.1.1 Bifurcation from a Simple Eigenvalue

Let us suppose that there exist $\lambda^* \in \mathbb{R}$ and $\varphi \in E \setminus \{0\}$ such that

$(L1)$ $Z := \mathrm{Ker}\,(L - \lambda^* I) = \mathbb{R}\varphi$;
$(L2)$ the codimension of $W = \mathrm{Range}\,(L - \lambda^* I)$ is one and $W = Z^\perp$.

Usually, this case is referred as the *simple eigenvalue case*. If $(L1)$–$(L2)$ hold then $E = Z \oplus W$ and any $u \in E$ can be written in a unique way as $u = \alpha\varphi + w$, with $\alpha \in \mathbb{R}$ and $w \in W$. Moreover, we can use the Lyapunov–Schmidt reduction (see Sect. 3.3) to write (6.1) as the system

$$\begin{cases} Lw + H(\alpha\varphi + w) = \lambda w, & \text{(auxiliary equation);} \\ (I - P)H(\alpha\varphi + w) = \lambda\alpha\varphi, & \text{(bifurcation equation).} \end{cases}$$

According to Lemma 3.3.1, the auxiliary equation has a unique solution $w(\lambda, \alpha) \in W$, defined in a neighborhood $\Lambda \times U \subset \mathbb{R} \times \mathbb{R}$ of $(\lambda^*, 0) \in \mathbb{R} \times \mathbb{R}$. Substituting into the bifurcation equation, we have to look for solutions of

$$(I - P)H(\alpha\varphi + w(\lambda, \alpha)) - \lambda\alpha\varphi = 0,$$

which is equivalent to

$$S(\lambda, \alpha) := (H(\alpha\varphi + w(\lambda, \alpha)) \mid \varphi) - \lambda\alpha = 0. \tag{6.2}$$

Suppose that $H \in C^2(E, E)$. Then the function $S : \Lambda \times U \to \mathbb{R}$ is of class $C^2$ and, according to the properties stated in Lemma 3.3.1, $S(\lambda, 0) = 0$ holds, for all $\lambda \in \Lambda$ and $S_\alpha(\lambda^*, 0) = 0$. In order to de-singularize $S$, we introduce the function

$$\sigma(\lambda, \alpha) = \begin{cases} \dfrac{S(\lambda, \alpha)}{\alpha} & \text{if } \alpha \neq 0; \\ S_\alpha(\lambda, 0) & \text{if } \alpha = 0. \end{cases}$$

The function $\sigma$ is of class $C^1$ in $\Lambda \times U$ and $\sigma(\lambda^*, 0) = S_\alpha(\lambda^*, 0) = 0$. Moreover, one has

$$\sigma_\lambda(\lambda^*, 0) = \lim_{\alpha \to 0} \frac{S_\lambda(\lambda^*, \alpha)}{\alpha}.$$

Since $H'(0) = 0$, a straight calculation shows:

$$S_\lambda(\lambda^*, \alpha) = (H'(0)[w_\lambda(\lambda^*, 0)] \mid \varphi) - \alpha = -\alpha,$$

and hence $\sigma_\lambda(\lambda^*, 0) = -1$. Then we can apply the implicit function Theorem 3.2.1 to $\sigma(\lambda, \alpha) = 0$, yielding $\lambda = \lambda(\alpha)$ such that $\sigma(\lambda(\alpha), \alpha) = 0$ for all $\alpha$ in a neighborhood of $\alpha = 0$. It is clear that the family $(\lambda_\alpha, u_\alpha) := (\lambda(\alpha), \alpha\varphi + w(\lambda(\alpha), \alpha))$ is a solution set of (6.1). Moreover, from $(u_\alpha \mid \varphi) = \alpha$ it follows that $u_\alpha \neq 0$ provided $\alpha \neq 0$. In conclusion, we have proved the following.

**Theorem 6.1.2** *Let $L : E \to E$ be a linear compact operator and suppose that $\lambda^*$ is a simple eigenvalue of $L$. Moreover, let $H \in C^2(E, E)$ be such that $H(0) = 0$, $H'(0) = 0$. Then $\lambda^*$ is a bifurcation point of* (6.1). *More precisely, the bifurcation set $\Sigma$ is, near $(\lambda^*, 0)$, a continuous curve.* □

The preceding result is a particular case of a more general one, dealing with the bifurcation for an equation as $F(\lambda, u) = 0$, where $F : \mathbb{R} \times X \to Y$, and $X, Y$ are Banach spaces. Here we will limit ourselves to state a result, referring for more details to, e.g., [16, Chap. 5] or [15, Chap. 2].

Let $F \in C^2(\mathbb{R} \times X, Y)$ be such that $F(\lambda, 0) \equiv 0$. We suppose that there exists $\lambda^* \in \mathbb{R}$, $\varphi \in X$ such that

$(F.1)$ Ker $[d_u F(\lambda^*, 0)] = span\{\varphi\}$.

Moreover, let $Y_0 \subset Y$ denote the range of $d_u F(\lambda^*, 0) \in L(X, Y)$ and assume

$(F.2)$ $Y_0$ is closed and its codimension is 1,
$(F.3)$ $d_{\lambda,u} F(\lambda^*, 0)[\varphi] \notin Y_0$.

Let us point out that, in the case in which $X = Y = E$ and $F(\lambda, u) = Lu - \lambda u + H(u)$, $(L1)$ is nothing but $(F.1)$. Moreover, $Y_0 = (\text{Ker}\,[d_u F(\lambda^*, 0)])^\perp$ and $d_{\lambda,u} F(\lambda^*, 0)[\varphi] = -\varphi$, and hence $(L2)$ is nothing but $(F.2)$–$(F.3)$.

**Theorem 6.1.3** *Let $F \in C^2(X, Y)$ be such that $F(\lambda, 0) \equiv 0$ and assume that $(F.1)$–$(F.3)$ hold. Then $\lambda = \lambda^*$ is a bifurcation point for $F(\lambda, u) = 0$. Precisely, from $(\lambda^*, 0)$ branches off a curve of nontrivial solutions of $F(\lambda, u) = 0$.* □

### *6.1.2 Bifurcation from an Odd Eigenvalue*

In this subsection we will give a theorem due to Krasnoselskii which deals with the case that $L$ and $H$ are compact. We use the same notation as in Theorem 6.1.2. It is also understood that $H(0) = 0$ and $H'(0) = 0$.

**Theorem 6.1.4** *Suppose that $L$ and $H$ are compact $C^1$ operators in a Banach space $X$ with $H(0) = 0$ and $H'(0) = 0$ and let $\lambda^*$ be an eigenvalue of $L$ with odd finite (algebraic) multiplicity. Then $\lambda^*$ is a bifurcation point for $Lu + H(u) = \lambda u$.*

*Proof* Let us remark that $\lambda^* \neq 0$ is an isolated eigenvalue of $L$. Setting

$$\Phi(\lambda, u) = Lu + H(u) - \lambda u,$$

one has that $\Phi_u(\lambda, 0) = L - \lambda I$. Then there exists $\varepsilon_0 > 0$ such that the unique eigenvalue of $L$ contained in the interval $[\lambda^* - \varepsilon_0, \lambda^* + \varepsilon_0]$ is $\lambda^*$. In particular, we can evaluate the Leray–Schauder index of $u = 0$ by linearization (see Theorem 4.2.11) yielding

$$i(\Phi(\lambda^* - \varepsilon, 0), 0) = i(L - (\lambda^* - \varepsilon)I, 0) = (-1)^{\beta_1}, \quad \forall\, 0 < \varepsilon \le \varepsilon_0,$$

where $\beta_1$ is the sum of the algebraic multiplicity mult($\lambda$) of all eigenvalues $\lambda$ with $\lambda \le \lambda^* - \varepsilon$.

Similarly,

$$i(\Phi(\lambda^* + \varepsilon, 0), 0) = i(L - (\lambda^* + \varepsilon)I, 0) = (-1)^{\beta_2}, \quad \forall\, 0 < \varepsilon \le \varepsilon_0,$$

where $\beta_2 = \sum_{\lambda \le \lambda^* + \varepsilon} \mathrm{mult}(\lambda)$. Then $\beta_2 = \beta_1 + \mathrm{mult}(\lambda^*)$ and, since $\lambda^*$ has odd algebraic multiplicity, it follows that

$$i(\Phi(\lambda^* + \varepsilon, 0), 0) = -i(\Phi(\lambda^* - \varepsilon, 0), 0). \tag{6.3}$$

On the other hand, if by contradiction $\lambda^*$ is not a bifurcation point, then there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for all $\varepsilon \in (0, \varepsilon_1)$ there holds

$$\Phi(\lambda, u) \ne 0, \quad \forall\, \lambda \in [\lambda^* - \varepsilon, \lambda^* + \varepsilon], \quad \forall\, \|u\| = \varepsilon. \tag{6.4}$$

This immediately implies that

$$i(\Phi(\lambda^* + \varepsilon, 0), 0) = i(\Phi(\lambda^* - \varepsilon, 0), 0),$$

a contradiction with (6.3), proving the theorem. □

*Remark 6.1.5* The above proof highlights that, even if $L$ and $H$ are only continuous and compact operators, $\lambda^*$ is a bifurcation point of the equation $\Phi(\lambda, u) = 0$ provided that there exists a change of the index of $i(\Phi(\lambda, 0), 0)$ as $\lambda$ crosses $\lambda = \lambda^*$.

## 6.2 Bifurcation for Variational Operators

In this section we will suppose that $L$ and $H$ are variational operators in a Hilbert space $E$, namely:

($A_1$) $L \in L(E, E)$ is a symmetric Fredholm operator with index zero and there exists a functional $\mathcal{H} \in C^k(E, \mathbb{R})$, for some $k \ge 3$, such that $H(u) = \mathcal{H}'(u)$. Moreover, $\mathcal{H}(0) = \mathcal{H}'(0) = \mathcal{H}''(0) = 0$.

Let us define $\mathcal{J} \in C^k(E, \mathbb{R})$ by setting

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\lambda\|u\|^2 - \frac{1}{2}(Lu \mid u) - \mathcal{H}(u). \tag{6.5}$$

It follows that $\mathcal{J}_\lambda'(u) = \lambda u - Lu - H(u)$. Let $\Sigma$ be the closure of $\{(\lambda, u) \in \mathbb{R} \times X : Lu + H(u) = \lambda u,\ u \ne 0\}$. Then, in this case, $\Sigma$ is the closure of the set of the critical points $u$ of $\mathcal{J}_\lambda$ on $E$ such that $u \ne 0$.

### 6.2.1 A Krasnoselskii Theorem for Variational Operators

**Theorem 6.2.1** (Krasnoselskii [60]) *Suppose that* $(A_1)$ *holds and let* $\lambda^*$ *be an isolated eigenvalue of* $L$ *with finite multiplicity. Then* $\lambda^*$ *is a bifurcation point of* (6.1).

*Proof* The proof will be divided into several steps. □

*Step 1. Lyapunov–Schmidt reduction.* Setting $Z = \mathrm{Ker}\,[\lambda^* I - L]$, $E = Z \oplus W$, $u = z + w$, $z \in Z$, $w \in W$ and letting $P$ denote the orthogonal projection on $W$, parallel to $Z$, the equation $\mathcal{J}'_\lambda(u) = 0$ splits into the system given by the auxiliary equation $P\mathcal{J}'_\lambda(z+w) = 0$ and the bifurcation equation $(I - P)\mathcal{J}'_\lambda(z+w) = 0$. Lemma 3.3.1 yields the existence of $w = w(\lambda, z)$ defined in a neighborhood $\mathcal{O}$ of $(\lambda^*, 0)$ in $\mathbb{R} \times Z$ such that $w(\lambda, 0) \equiv 0$ and $P\mathcal{J}'_\lambda(z+w(\lambda, z)) = 0$. Moreover (see Remark 3.3.2), $w \in C^{k-1}(\mathcal{O}, W)$ and $d_z^j w(\lambda^*, 0) = 0\ \forall\, j = 1, \ldots, k-2$. In particular,

$$\|w(\lambda, z)\| \le \|z\|, \quad \forall\, (\lambda, z) \in \mathcal{O}, \tag{6.6}$$

uniformly with respect to $\lambda$.

*Step 2. Study of the bifurcation equation.* Substituting $w(\lambda, z)$ into the bifurcation equation, we are led to find $z \in Z$ such that

$$(I - P)\mathcal{J}'_\lambda(z + w(\lambda, z)) = 0. \tag{6.7}$$

To solve Eq. (6.7) we will take advantage of the fact that we are in the variational case. Let us define $\mathcal{J}_\lambda : Z \to \mathbb{R}$ by setting

$$\mathcal{J}_\lambda(z) = \mathcal{J}_\lambda(z + w(\lambda, z)).$$

**Lemma 6.2.2** *If* $z_\lambda \in Z$ *is a critical point of* $\mathcal{J}_\lambda$ *then* $u_\lambda = z_\lambda + w(\lambda, z_\lambda)$ *is a solution of* (6.1). *Furthermore, if* $z_\lambda \neq 0$ *and* $\|z_\lambda\| \to 0$ *as* $|\lambda| \to \lambda^*$, *then* $u_\lambda \neq 0$ *and* $\|u_\lambda\| \to 0$.

*Proof* If $z_\lambda \in Z$ is a critical point of $\mathcal{J}_\lambda$ there results

$$(\mathcal{J}'_\lambda(u_\lambda) \mid \zeta + d_z w(\lambda, z_\lambda)[\zeta]) = 0, \quad \forall\, \zeta \in Z.$$

Le us remark that $P\mathcal{J}'_\lambda(z + w(\lambda, z)) = 0$ for all $z \in Z$. In particular, $P\mathcal{J}'_\lambda(u_\lambda) = 0$, namely $\mathcal{J}'_\lambda(u_\lambda) \in Z$. Since $d_z w(\lambda, z_\lambda)[\zeta] \in W$ we infer

$$(\mathcal{J}'_\lambda(u_\lambda) \mid d_z w(\lambda, z_\lambda)[\zeta]) = 0, \quad \forall\, \zeta \in Z.$$

Thus $(\mathcal{J}'_\lambda(u_\lambda) \mid \zeta) = 0$, for all $\zeta \in Z$. Using again the fact that $P\mathcal{J}'_\lambda(u_\lambda) = 0$ we conclude that $\mathcal{J}'_\lambda(u_\lambda) = 0$. The second part of the lemma is deduced by using the fact that $d_z w(\lambda, 0) = 0$. □

*Step 3. Finding nontrivial critical points of* $\mathcal{J}_\lambda$ *on* $Z$. In order to find a nontrivial critical point of $\mathcal{J}_\lambda$ on $Z$, we will make, for the reader's convenience, some additional assumptions that will simplify the arguments. Specifically, we will suppose

$(A_2)$ there is an integer $k \geq 3$ such that $d^j\mathcal{H}(0) = 0$, $\forall j = 1, \dots, k-1$, and $d^k\mathcal{H}(0) \neq 0$. Let

$$\alpha_k(v) = \frac{1}{k!} d^k\mathcal{H}(0)[v]^k, \quad v \in E.$$

$(A_3)$ The maximum $M$ and the minimum $m$ of $\alpha_k$ in the boundary of the unit ball in $Z$ have the same sign: either $M \geq m > 0$ or $m \leq M < 0$.

Let us point out that the function $\alpha_k$ is homogeneous of degree $k$ and hence $(A_3)$ is always satisfied if $k$ is even. Furthermore, there results

$$\mathcal{H}(u) = \alpha_k(u) + o(\|u\|^k) \quad \text{as } \|u\| \to 0.$$

Let us evaluate $\mathcal{J}_\lambda(z)$. For brevity we write $w$ instead of $w(\lambda, z)$. One has that

$$(L(z+w) \mid z+w) = (Lz \mid z) + (Lw \mid w) = \lambda^*\|z\|^2 + (Lw \mid w)$$

and hence

$$\mathcal{J}_\lambda(z) = \frac{\lambda - \lambda^*}{2}\|z\|^2 + \frac{\lambda}{2}\|w\|^2 - \frac{1}{2}(Lw \mid w) - \mathcal{H}(z+w).$$

Let us remark that $w$ satisfies $P\mathcal{J}'_\lambda(z+w) = 0$. Using the specific form of $\mathcal{J}_\lambda$, one has that $P\mathcal{J}'_\lambda(u) = \lambda Pu - LPu - PH(u)$ and the equation $P\mathcal{J}'_\lambda(z+w) = 0$ becomes $\lambda w - Lw = PH(z+w)$. This implies

$$\lambda\|w\|^2 - (Lw \mid w) = (H(z+w) \mid w),$$

and therefore

$$\mathcal{J}_\lambda(z) = \frac{\lambda - \lambda^*}{2}\|z\|^2 + \frac{1}{2}(H(z+w) \mid w) - \mathcal{H}(z+w).$$

Moreover, for some $s \in (0, 1)$,

$$\mathcal{H}(z+w) = \mathcal{H}(z) + (H(z+sw) \mid w).$$

Hence we find

$$\mathcal{J}_\lambda(z) = \frac{\lambda - \lambda^*}{2}\|z\|^2 - \mathcal{H}(z) + \frac{1}{2}(H(z+w) \mid w) - (H(z+sw) \mid w). \qquad (6.8)$$

We now estimate the last three terms in (6.8). Let $M \geq m > 0$ (if $m \leq M < 0$, we simply consider $-\mathcal{J}_\lambda$) and let $\lambda^* < m/(1+2^k)$. Since $\mathcal{H}'(u) = H(u)$ and $d^j\mathcal{H}(0) = 0$, for all $j \le k-1$, there exists $\rho > 0$, depending on $\lambda$, such that

$$\|H(u)\| \le \lambda^* \|u\|^{k-1}, \quad \forall \|u\| < \rho,$$

and hence

$$\mathcal{H}(z) = \alpha_k(z) + \beta(z), \quad |\beta(z)| \le \lambda^* \|z\|^k, \ \forall \|z\| < \rho.$$

The estimate (6.6) implies that for all $r < \rho/2$ there exists $\varepsilon_0 > 0$ such that

$$\|z + w(\lambda, z)\| \le \|z\| + \|w(\lambda, z)\| \le 2\|z\| < 2r < \rho$$

and this yields

$$\|H(z + w(\lambda, z))\| \le \lambda^* 2^{k-1} \, \|z\|^{k-1}, \quad \forall \ \|z\| < r, \ \ \forall \ |\lambda - \lambda^*| < \varepsilon_0.$$

This implies

$$|(H(z+w) \mid w)| \le \|H(z+w)\| \, \|w\| \le \lambda^* 2^{k-1} \, \|z\|^k, \quad \forall \ \|z\| < r, \ \forall \ |\lambda - \lambda^*| < \varepsilon_0,$$

and

$$\mathcal{H}(z) = \alpha_k(z) + \beta(z), \quad |\beta(z)| \le \lambda^* \|z\|^k, \ \forall \ \|z\| < \rho.$$

In conclusion, we can state the following lemma.

**Lemma 6.2.3** *Given $\lambda^* < m/(1+2^k)$ there exist $r > 0$ and $\varepsilon_0 > 0$ such that*

$$\mathcal{J}_\lambda(z) = \frac{\lambda - \lambda^*}{2} \|z\|^2 - \alpha_k(z) + R(\lambda, z), \tag{6.9}$$

*where*

$$R(\lambda, z) = \frac{1}{2}(H(z+w) \mid w) - (H(z+sw) \mid w) + \beta(z)$$

*satisfies*

$$|R(\lambda, z)| \le \lambda^* 2^k \, \|z\|^k + \lambda^* \, \|z\|^k, \quad \forall \ \|z\| < r, \ \ \forall \ |\lambda - \lambda^*| < \varepsilon_0. \tag{6.10}$$

*Step 4.* We are finally in position to prove that $\mathcal{J}_\lambda$ has a mountain pass critical point provided $|\lambda - \lambda^*|$ is small. We will assume that $\lambda - \lambda^* > 0$; if $\lambda - \lambda^* < 0$ we consider $-\mathcal{J}_\lambda$ and argue in the same way. First, some further preliminaries are in order.

Let $z \in Z$ be such that $\|z\| < r$ and let $0 < \lambda - \lambda^* < \varepsilon_0$. Using (6.9), the inequality $\mathcal{J}_\lambda(z) > 0$ implies

$$\frac{\lambda - \lambda^*}{2} \|z\|^2 > \alpha_k(z) - R(\lambda, z).$$

Then (6.10) and $\alpha_k(z) \ge m\|z\|^k$ yield

$$\frac{\lambda - \lambda^*}{2} \|z\|^2 < m\|z\|^k - \lambda^*(1+2^k)\|z\|^k = [m - \lambda^*(1+2^k)]\|z\|^k.$$

Since $m > \lambda^*(1+2^k)$ and $k \ge 3$ it follows that there exists $0 < \varepsilon' < \varepsilon_0$ such that

$$\|z\| < r_\lambda := \left[ \frac{\lambda - \lambda^*}{2(m - \lambda^*(1+2^k))} \right]^{1/(k-2)} < r. \tag{6.11}$$

It follows that for $0 < \lambda - \lambda^* < \varepsilon'$ the following holds:

$$\max_{\|z\|=r} \mathcal{J}_\lambda(z) < 0.$$

This allows us to define $\widetilde{\mathcal{J}}_\lambda : Z \to \mathbb{R}$ in such a way that

(1) $\widetilde{\mathcal{J}}_\lambda(z) = \mathcal{J}_\lambda(z)$ for all $\|z\| \le r$;
(2) $\widetilde{\mathcal{J}}_\lambda \in C^1(Z, \mathbb{R})$;
(3) $\widetilde{\mathcal{J}}_\lambda(z) < 0$ for all $\|z\| \ge r$.

Since $\widetilde{\mathcal{J}}_\lambda(z) = \mathcal{J}_\lambda(z)$ for all $\|z\| \le r$, it immediately follows from (6.9) and (6.10) that $\widetilde{\mathcal{J}}_\lambda$ has a local strict minimum at $z = 0$, namely that $(J1)$ of Chap. 5 holds. Obviously $\widetilde{\mathcal{J}}_\lambda$ satisfies $(J2)$. Furthermore, from 3) above, it follows that any sequence $z_n \in Z$, such that $\widetilde{\mathcal{J}}_\lambda(z_n) \to c > 0$, is bounded. Since $Z$ is finite dimensional, it follows that $\widetilde{\mathcal{J}}_\lambda$ satisfies the $(PS)_c$. Applying the mountain pass theorem to $\widetilde{\mathcal{J}}_\lambda$, we find $z_\lambda \in Z$ such that $\widetilde{\mathcal{J}}'_\lambda(z_\lambda) = 0$ and $\widetilde{\mathcal{J}}_\lambda(z_\lambda) = c > 0$. From the latter we infer that $\|z_\lambda\| < r$ therefore $\widetilde{\mathcal{J}}_\lambda(z_\lambda) = \mathcal{J}_\lambda(z_\lambda)$. More precisely, it follows from (6.11) that $\|z_\lambda\| < r_\lambda$ and hence $\|z_\lambda\| \ne 0$ is such that $\|z_\lambda\| \to 0$ as $\lambda \to \lambda^*$. Using Lemma 6.2.2, Theorem 6.2.1 follows. □

*Remark 6.2.4* An elegant proof of Theorem 6.2.1 has been given by Marino and Prodi [68] by using the Morse theory.

### 6.2.2 *Branching Points*

In applications it is important to know whether a *branch* of solutions emanates from a bifurcation point. Precisely, we say that $\lambda^*$ is a *branching point* of (6.1) if the solution set $\Sigma$ contains a connected set $\mathcal{S}$ such that $(\lambda^*, 0) \in \mathcal{S}$ and $\mathcal{S} \setminus \{(\lambda^*, 0)\} \ne \emptyset$. For example, as we will see in Theorem 6.3.1, $\lambda^*$ is a branching point provided $\lambda^*$ is an eigenvalue of $L$ with finite odd multiplicity. Actually, if $\lambda^*$ is a simple eigenvalue of $L$, by Theorem 6.1.2, the set $\mathcal{S}$ in $\Sigma$ is, in a neighborhood of $(\lambda^*, 0)$, a curve. On the other hand, in the general case of non-necessarily odd multiplicity, $\lambda^*$ might not be a branching point. Bhöme [31] has given an example of a variational problem in $\mathbb{R}^2$ where $\mathcal{H} \not\equiv 0$ is $C^\infty$ with all the derivatives at $u = 0$ equal to zero and $\lambda^*$ is not a branching point. It is worth pointing out explicitly that in the Bhöme example condition $(A_2)$ is not satisfied. The interested reader may see [15].

In order to prove the existence of a branching point, we will assume, in addition to $(A_1)$ and $(A_2)$, the following condition.

$(A_4)$ Let $\xi \in \partial B_Z$, resp. $\eta \in \partial B_Z$, be such that $\alpha_k(\xi) = M$, resp. $\alpha_k(\eta) = m$. We assume that $kM$ and $km$ are not eigenvalues of the matrix $D^2\alpha_k(\xi)$, resp. $D^2\alpha_k(\eta)$.

The following theorem is proved in [6], to which we refer for more details and further results dealing with the existence of branching points for (6.1).

**Theorem 6.2.5** *Suppose that* $(A_1, A_2)$ *and* $(A_4)$ *hold and let* $\lambda^*$ *be an isolated eigenvalue of finite multiplicity of* $L$. *Then* $\lambda^*$ *is a branching point of* (6.1). □

*Remark 6.2.6* $(A_4)$ rules out the functions $\alpha_k$ such that $\alpha_k(z) \equiv c\|z\|^k$ on $Z$. If this is violated, there are examples showing that $\lambda^*$ can be a bifurcation point but not a branching point (see [6]).

We will not give the proof of Theorem 6.2.5 here, but we will merely highlight the role of assumption $(A_4)$.

Setting $\varepsilon = \lambda - \lambda^*$ and

$$\Psi_\varepsilon(z) = \frac{1}{2}\,\varepsilon\,\|z\|^2 - \alpha_k(z), \quad z \in Z,$$

the auxiliary functional $\mathcal{J}_\varepsilon$ can be written in the form

$$\mathcal{J}_\varepsilon(z) = \Psi_\varepsilon(z) + R(\varepsilon, z).$$

The functional $\Psi_\varepsilon$ has the mountain pass geometry. However, in this case it is convenient to find the mountain pass critical point in a more direct way. Since $\alpha_k \not\equiv 0$, if $T = \{z \in Z : \|z\| = 1\}$, then either $M := \max_T \alpha_k > 0$ or $\min_T \alpha_k < 0$. Assume the former: in the other case it suffices to consider $-\varepsilon$ instead of $\varepsilon$. Let $\xi \in T$ be a point where M is achieved. By homogeneity it immediately follows that $\alpha_k'(\xi) = k\alpha(\xi)\xi = kM\xi$. Moreover, $p_\varepsilon = t_\varepsilon \xi$ is a critical point of $\Psi_\varepsilon$ whenever $t_\varepsilon$ satisfies

$$t_\varepsilon^{k-2} = \frac{\varepsilon}{kM} \quad (\varepsilon > 0). \tag{6.12}$$

It is easy to check that $p_\varepsilon$ is the mountain pass critical point of $\Psi_\varepsilon$ we were seeking. Next, using $(A_4)$ one can show that $p_\varepsilon$ is a non-degenerate mountain pass critical point of $\Psi_\varepsilon$ and there results

$$i(\Gamma'_\varepsilon, p_\varepsilon) = -1. \tag{6.13}$$

Roughly, let $I_Z$ denote the identity in $Z$ and let $A_k = D^2\alpha_k$. Then

$$D^2\Psi_\varepsilon(p_\varepsilon) = \varepsilon I_Z - A_k(p_\varepsilon).$$

Since $p_\varepsilon = t_\varepsilon \xi$ and using (6.12), one finds that

$$D^2\Psi_\varepsilon(p_\varepsilon) = \varepsilon I_Z - t_\varepsilon^{k-2} A_k(\xi) = \varepsilon I_Z - \frac{\varepsilon}{kM} A_k(\xi).$$

By $(A_4)$, $kM$ is not an eigenvalue of $A_k(\xi)$. Hence $D^2\Psi_\varepsilon(p_\varepsilon)$ is invertible and $p_\varepsilon$ is a non-degenerate critical point of $\Psi_\varepsilon$. As $p_\varepsilon$ is a non-degenerate mountain pass critical point, it is well known that (6.13) holds; see Remark 5.3.7-(ii). Since $R$ satisfies

(6.10), the properties of the topological degree imply that for $\varepsilon > 0$ sufficiently small one also has

$$\deg(\mathcal{J}'_\varepsilon, B(p_\varepsilon, \delta), 0) = -1, \quad \delta > 0 \text{ small},$$

where $B(p_\varepsilon, \delta)$ denote a ball in $Z$ centered in $p_\varepsilon$ with radius $\delta$. At this point, the properties of the degree and an appropriate limiting procedure allow us to prove that the bifurcation set $\Sigma$ contains a connected set $\mathcal{S}$ and hence $\lambda^*$ is a branching point.

It is also possible to give a more precise description of $\mathcal{S}$. We set $\Pi(\mathcal{S}) = \{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{S}\}$. Under the same assumptions of Theorem 6.2.1 one can show:

(i) If $k$ is odd then $\Pi(\mathcal{S})$ contains an interval $[a, b]$ such that $a < \lambda^* < b$.
(ii) If $k$ is even then $\Pi(\mathcal{S})$ contains a one-sided neighborhood $\Lambda$ of $\lambda^*$ such that for all $\lambda \in \Lambda \setminus \{\lambda^*\}$ (6.1) has at least two distinct nontrivial solutions on $\mathcal{S}$. Furthermore, if $d = \dim(Z) \geq 2$ and $\alpha_k(z) > 0$ (resp. $< 0$) for all $z \in Z \setminus \{0\}$, then for every $\lambda = \lambda^* + \varepsilon$, with $\varepsilon > 0$ (resp. $\varepsilon < 0$) sufficiently small, (6.1) possesses at least two pairs of distinct solutions on $\mathcal{S}$.

## 6.3 Global Bifurcation

P. Rabinowitz [74] improved Theorem 6.1.4 by showing that, under the same hypotheses, the continuum $\mathcal{S}$ of $\Sigma$ which contains $(\lambda^*, 0)$ is either unbounded or contains another bifurcation point $\lambda^\sharp \neq \lambda^*$. By Remark 6.1.5, we explicitly state the result under the hypothesis that there is a change of index of

$$\Phi_\lambda(u) := \Phi(\lambda, u)$$

when we cross $\lambda = \lambda^*$, instead of assuming that $\lambda^*$ is an eigenvalue of the linear part of $\Phi_\lambda$ with odd finite multiplicity.

**Theorem 6.3.1** *Let $\lambda^* \in \mathbb{R}$ and $\varepsilon_0 > 0$ be such that the set $(\lambda^* - \varepsilon_0, \lambda^* + \varepsilon_0) \setminus \{\lambda^*\}$ does not contain bifurcation points of* (4.9$_\lambda$). *Assume also that for every $\underline{\lambda} \in (\lambda^* - \varepsilon_0, \lambda^*)$ and $\overline{\lambda} \in (\lambda^*, \lambda^* + \varepsilon_0)$ the following holds:*

$$\mathrm{i}(\Phi_{\underline{\lambda}}, 0) \neq \mathrm{i}(\Phi_{\overline{\lambda}}, 0). \tag{6.14}$$

*Then the connected component, $\mathcal{S}$, of $\Sigma$ that contains $(\lambda^*, 0)$ satisfies at least one of the following conditions:*

(i) *$\mathcal{S}$ is unbounded in $\mathbb{R} \times X$,*
(ii) *there exists a bifurcation point $\lambda^\sharp \in \mathbb{R} \setminus \{\lambda^*\}$ such that $(\lambda^\sharp, 0) \in \mathcal{S}$.*

*Proof* By Theorem 6.1.4 (see also Remark 6.1.5), $\lambda^*$ is a bifurcation point from zero of $\Phi_\lambda(u) = 0$. Let $\mathcal{S}$ be the connected component of $\Sigma$ which contains $(\lambda^*, 0)$.

We argue by contradiction and assume that $\mathcal{S}$ verifies neither (i) nor (ii). This means that $\mathcal{S}$ is bounded and that for every $\lambda \neq \lambda^*$ there exists $\rho(\lambda) > 0$ such that

$$\mathcal{S}_\lambda \cap B_{\rho(\lambda)}(0) = \emptyset.$$

We claim that there exists a bounded set $\mathcal{O} \subset \mathbb{R} \times X$ and $\varepsilon_0 > 0$ satisfying

$$\partial\mathcal{O} \cap \Sigma = \emptyset, \tag{6.15}$$

$$(\lambda^*, 0) \in \mathcal{O} \tag{6.16}$$

and

$$\mathcal{O} \cap (\mathbb{R} \times \{0\}) \subset (\lambda^* - \varepsilon_0, \lambda^* + \varepsilon_0) \times X. \tag{6.17}$$

Indeed, if $U_\delta$ denotes the neighborhood of $\mathcal{S}$ consisting in all points with distance to $\mathcal{S}$ less than $\delta$, then in the case $\Sigma \cap \partial U_\delta = \emptyset$ it suffices to take $\mathcal{O} = U_\delta$. In the other case, since the set $M = \overline{U_\delta} \cap \Sigma$ is a compact metric space, we can apply Lemma 4.3.1 to the closed sets $\mathcal{S}$ and $\Sigma \cap \partial U_\delta$ to conclude the existence of two compact, disjoint subsets $A, B$ of $M$, with

$$M = A \cup B,\ \mathcal{S} \subset A.$$

By taking as $\mathcal{O}$ a neighborhood of $A$ of all points with distance to $A$ less than the distance between $A$ and $B$, we obtain (6.15)–(6.16).

The general homotopy property allows to deduce then that

$$\deg(\Phi_\lambda, \mathcal{O}_\lambda, 0) = \text{ const.}, \quad \forall \lambda \in \mathbb{R}. \tag{6.18}$$

Now, we are going to compute this degree. To do it, fix $\overline{\lambda} \in (\lambda^*, \lambda^* + \varepsilon_0)$ such that $(\overline{\lambda}, 0) \in \mathcal{O}$. We can choose $\rho > 0$ such that:

(a) For every $\lambda \in [\overline{\lambda}, \lambda^* + \varepsilon_0]$, the problem $(4.9)_\lambda$ has no nontrivial solutions in $\overline{B_\rho(0)}$, i.e.,

$$\Sigma_\lambda \cap \overline{B_\rho(0)} = \emptyset.$$

(b) For every $\lambda \geq \lambda^* + \varepsilon_0$, the $\lambda$-slice $\mathcal{O}_\lambda$ of $\mathcal{O}$ does not contain points of the closed ball $\overline{B_\rho(0)}$, i.e.,

$$\mathcal{O}_\lambda \cap \overline{B_\rho(0)} = \emptyset.$$

Take

$$\mathcal{U} = \mathcal{O} \cap \left[[\overline{\lambda}, +\infty) \times (X \setminus \overline{B_\rho(0)})\right].$$

Observe that the $\lambda$-slice $\mathcal{U}_\lambda$ of $\mathcal{U}$ is given by

$$\mathcal{U}_\lambda = \mathcal{O}_\lambda \setminus \overline{B_\rho(0)},$$

for every $\lambda \geq \overline{\lambda}$.

By (a) and (b), the general homotopy property of the degree implies that

$$\deg(\Phi_\lambda, \mathcal{U}_\lambda, 0) = \text{ constant}, \quad \forall \lambda \geq \overline{\lambda}.$$

But, since $\mathcal{O}$ is bounded, $\mathcal{U}_\lambda = \mathcal{O}_\lambda \setminus \overline{B_\rho(0)} = \mathcal{O}_\lambda = \emptyset$ provided that $\lambda \gg \overline{\lambda}$. We obtain as a consequence that the above degree is zero. In particular, $\deg(\Phi_{\overline{\lambda}}, \mathcal{U}_{\overline{\lambda}}, 0) = 0$, that is,

$$\deg(\Phi_{\overline{\lambda}}, \mathcal{O}_{\overline{\lambda}} \setminus \overline{B_\rho(0)}, 0) = 0.$$

By the additivity property we conclude that

$$\begin{aligned}\deg(\Phi_{\overline{\lambda}}, \mathcal{O}_{\overline{\lambda}}, 0) &= \deg(\Phi_{\overline{\lambda}}, \mathcal{O}_{\overline{\lambda}} \setminus \overline{B_\rho(0)}, 0) + \deg(\Phi_{\overline{\lambda}}, B_\rho(0), 0)\\ &= \mathrm{i}(\Phi_{\overline{\lambda}}, 0).\end{aligned}$$

Similarly, if we fix $\underline{\lambda} \in (\lambda^* - \varepsilon_0, \lambda^*)$ such that $(\underline{\lambda}, 0) \in \mathcal{O}$, we can prove that

$$\deg(\Phi_{\underline{\lambda}}, \mathcal{O}_{\underline{\lambda}}, 0) = \mathrm{i}(\Phi_{\underline{\lambda}}, 0).$$

Consequently, taking into account (6.18) we conclude that

$$\mathrm{i}(\Phi_{\underline{\lambda}}, 0) = \mathrm{i}(\Phi_{\overline{\lambda}}, 0),$$

which contradicts (6.14). □

Now, as a direct consequence of the above theorem, we have the classical improvement by Rabinowitz of the theorem of Krasnoselskii.

**Corollary 6.3.2** *Under the hypotheses of Theorem 6.1.4, there exists a continuum $\mathcal{S}$ of $\Sigma$ that either is unbounded, or $(\lambda^\sharp, 0) \in \mathcal{S}$ for another eigenvalue $\lambda^\sharp \neq \lambda^*$ of $L$.* □

# Chapter 7
# Elliptic Problems and Functional Analysis

The purpose of this chapter is to show how a nonlinear elliptic problem can be transformed into an operator equation that can be treated with the abstract tools discussed in the previous chapters.

## 7.1 Nonlinear Elliptic Problems

The abstract results proved in the preceding chapters will be applied to elliptic problems such as

$$\begin{cases} -\Delta u = \lambda f(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. In the sequel we will assume that $f$ is sufficiently smooth. This will simplify the exposition, though in many cases weaker regularity assumptions could be made.

It turns out that weak solutions are classical solutions provided $f(x,u)$ is Hölder continuous and for some $p \in [1, +\infty)$ it satisfies the growth condition

$$|f(x,u)| \le c_1 + c_2|u|^p, \quad p < 2^* - 1, \tag{7.2}$$

where (see Notation) $2^*$ denotes the critical Sobolev exponent of $H_0^1(\Omega)$ in Theorem A.4.3, i.e.,

$$2^* = \begin{cases} \dfrac{2N}{N-2}, & \text{if } N \ge 3; \\ +\infty, & \text{if } N = 1, 2. \end{cases} \tag{7.3}$$

To prove this fact one uses what is called *bootstrap argument*. It is based on the following property: *if a weak solution $u \in H_0^1(\Omega)$ of (7.1) belongs to $L^r(\Omega)$ for some $r > 1$, then, by (7.2), $f(x,u(x)) \in L^{r/p}(\Omega)$ and the Agmon–Douglis–Nirenberg*

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82,
DOI 10.1007/978-0-8176-8114-2_7, © Springer Science+Business Media, LLC 2011

*estimates (Theorem 1.2.11-1) imply that* $u \in W^{2,\frac{r}{p}}(\Omega)$. By the Sobolev embedding (see Theorem 1.1.3-1, we can begin by taking $r_0 = 2^*$ to deduce that $u \in W^{2,\frac{2^*}{p}}(\Omega)$. Using again Theorem 1.1.3-1 we have three possible cases:

1. $u \in L^{\frac{N2^*}{pN-22^*}}(\Omega)$ (if $Np < 2^*\,2$).
2. $u \in L^t(\Omega)$ for every $t > 2$ (if $Np = 2^*\,2$), and applying again Theorem 1.2.11-1, we derive that $u \in W^{2,t}(\Omega)$. In this case, choosing $t > N/2$, we infer (by Theorem 1.1.3-1) that $u \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$.
3. $u \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ (provided $Np > 2^*\,2$).

Observe that in the first case $\frac{N2^*}{pN-22^*} > r_0$ and we can iterate the process by taking now $r_1 = \frac{N2^*}{pN-22^*}$. It is easy to prove that in a finite number of iterations we get $u \in W^{2,\frac{r_k}{p}}(\Omega)$ with $\frac{r_k}{p} > \frac{N}{2}$. Thus, also in this case we have $u \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$. Finally, since $f$ is Hölder, $f(x,u(x))$ is also Hölder and, by the Schauder estimates (Theorem 1.2.11-2), we conclude that $u \in C^2(\overline{\Omega})$.

Problem (7.1) can be transformed into an operator equation in several ways, depending on the abstract tools we are going to use.

### 7.1.1 Classical Formulation

If we are working with the classical formulation of the problem, then, for example, we can let $X = \{u \in C^2(\overline{\Omega}) : u(x) = 0,\ \forall x \in \partial\Omega\}$, $Y = C(\overline{\Omega})$ and $T_\lambda(u) = \Delta u + \lambda f(x,u)$. Then any solution $u \in X$ of the equation $T_\lambda(u) = 0$ is a solution of (7.1). This framework is well suited for the use of the local inversion theorem or the implicit function theorem.

In order to use topological degree theoretic arguments, we could take for instance either $X = \{u \in C^{0,\nu}(\overline{\Omega}) : u(x) = 0, \forall x \in \partial\Omega\}$, $0 \le \nu < 1$, or $X = C_0^1(\overline{\Omega})$ and consider the operator $K$ introduced in Sect. 1.2.5, i.e., $w = Ku$ is the unique solution of $-\Delta w = u$ in $\Omega$ satisfying $w_{|\partial\Omega} = 0$. Let us point out that the Nemitski operator $f$, i.e., the operator which maps every function $u \in X$ into the function $f \circ u$, is continuous on $X$. Setting $T(u) = Kf(u)$ and $\Phi_\lambda(u) = u - \lambda T(u)$, the solutions of $\Phi_\lambda(u) = 0$ correspond to solutions of (7.1). Moreover, for any $u \in X$ one has that $w = Ku \in C^{2,\nu}(\overline{\Omega})$, and Ascoli's theorem implies that $K$ maps bounded sets in relatively compact sets in $X$. As a consequence, the nonlinear operator $T : X \longrightarrow X$ is compact. Hence $\Phi$ is a compact perturbation of the identity and we can employ the Schauder fixed point theorem or else the homotopy invariance of the degree. The reader has to observe that the application of these tools requires us to prove *a priori bounds*. By this one means that there exists $M > 0$ such that $\|u\| \le M$ for any possible solution of (7.1).

### 7.1.2 Weak Formulation

On the other hand, if we are working with the weak formulation of the problem (7.5) and we want to employ critical point theory, it is convenient to work on a

Hilbert space. Usually, one chooses the Sobolev space $E = H_0^1(\Omega)$ endowed with the scalar product and norm

$$(u \mid v) = \int \nabla u \cdot \nabla v\,, \quad \|u\| = \left(\int |\nabla u|^2\right)^{1/2}.$$

Let $F(x,u) = \int_0^u f(x,s)ds$ with $f$ satisfying (7.2) and define the functional $\mathcal{J}_\lambda : E \to \mathbb{R}$ by setting

$$\mathcal{J}_\lambda(u) = \tfrac{1}{2}\|u\|^2 - \lambda \int F(x,u(x))\,.$$

Observe that $F(x,u(x))$ is integrable by (7.2). In addition,

$$d\mathcal{J}_\lambda(u)[v] = (u \mid v) - \lambda \int f(x,u)v = \int \nabla u \cdot \nabla v - \lambda \int f(x,u)v.$$

Then any critical point $u \in E$ of $\mathcal{J}_\lambda$ verifies

$$\int \nabla u \cdot \nabla v \;-\lambda \int f(x,u)v = 0, \quad \forall\, v \in E,$$

and hence is a weak solution of (7.1). Since $f$ is smooth, by elliptic regularity, $u$ is a classical solution. It is instructive to evaluate the gradient $\mathcal{J}_\lambda'(u)$. By definition, $\mathcal{J}_\lambda'(u) \in E$ is such that $d\mathcal{J}_\lambda(u)[v] = (J_\lambda'(u) \mid v)$, for all $v \in E$, namely

$$\int \nabla u \cdot \nabla v \;-\lambda \int f(x,u)v = \int \nabla z \cdot \nabla v\,, \quad \forall\, v \in E,$$

where $z = \mathcal{J}_\lambda'(u)$. Setting $w = u - z$, then $w$ is such that

$$\int \nabla w \cdot \nabla v \;= \lambda \int f(x,u)v\,, \quad \forall\, v \in E.$$

Therefore $w$ is a solution of $-\Delta w = \lambda f(x,u)$, with $w_{|\partial\Omega} = 0$. In other words, $w = \lambda K \circ f(u)$ where $K$ denotes the inverse of the operator $-\Delta$ on $H_0^1(\Omega)$. In conclusion we have found that

$$\mathcal{J}_\lambda'(u) = u - \lambda K \circ f(u).$$

**Lemma 7.1.1** *The functional $\mathcal{J}_\lambda$ satisfies the compactness part* (b) *the (PS) condition (see Section 5.3).*

*Proof* Indeed, let $\{u_n\}$ be a bounded sequence in $H_0^1(\Omega)$ such that $\mathcal{J}_\lambda(u_n)$ is bounded and $\mathcal{J}_\lambda{}'(u_n) \longrightarrow 0$. Taking $u_n - u$ as a test function, we obtain $\mathcal{J}_\lambda{}'(u_n)(u_n - u) \longrightarrow 0$, i.e.,

$$(u_n \mid u_n - u) - \lambda(K(f(u_n)) \mid u_n - u) \longrightarrow 0.$$

By (7.2) and the compactness of $K$, we infer that $K(f(u_n))$ strongly converges to $K(f(u))$ up to a subsequence. Then $(K(f(u_n)) \mid u_n - u)$ converges to zero and hence $(u_n \mid u_n - u)$ tends to zero. This immediately implies that $u_n$ is strongly convergent. □

In some cases one can look for pairs $(\lambda, u)$ in $\mathbb{R} \times E$ satisfying (7.1) with prescribed norm of $u$, i.e., such that $\|u\| = R$, for some fixed $R > 0$. We will refer to this problem as a *nonlinear eigenvalue problem*. The difference with respect to the boundary value problem (7.1) is that here $\lambda$ is not a given number, but is unknown and appears as a Lagrange multiplier. In order to find solutions of such an eigenvalue problem, it is natural to use variational methods, in particular, Theorem 5.4.4 minimizing $\mathcal{F}(u) = \int F(x, u(x))dx$ on the sphere $\|u\| = R$. According to Example 5.2.2, a local minimum of such an $\mathcal{F}$ on the sphere satisfies $\mathcal{F}'(u) = \lambda u$ for a suitable Lagrange multiplier $\lambda \in \mathbb{R}$. As we have seen before, $u$ is a solution of (7.1) such that $\|u\| = R$.

On the other hand, if we want to apply bifurcation theory to study the existence of weak solutions of the boundary value problem

$$\begin{cases} -\Delta u = \lambda u + f(x, u), & \text{in } \Omega \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7.4}$$

we set $E = L^2(\Omega)$ and let $K$ be the inverse of the Laplacian operator (see Sect. 1.2.5), so that (7.4) is equivalent to $u = \lambda K u + K f(u)$, $u \in E$. Setting $L = K$ and $H = K f$, we get $H(0) = 0$ and $H'(0) = 0$ provided $f(x, 0) = 0$ and $\frac{\partial f}{\partial u}(x, 0) = 0$. Hence, we are in the abstract setting discussed in Chap. 6 dealing with bifurcation theory.

In this case the possible bifurcation points are the *characteristic values* $\lambda$ of $K$, i.e., the $\lambda$ such that $\operatorname{Ker}(\lambda K - I) \neq \{0\}$ or, equivalently, the eigenvalues of $-\Delta$ (see Lemma 6.1.1).

Let us remark that from elliptic regularity and the Rellich theorem (see Sect. 1.2.5) it follows that $K$ is a compact operator.

*Remark 7.1.2* It is worth pointing out that one can use other ways to frame the boundary value problems (7.4) or (7.1). For example, we can find bifurcation results by means of degree theory (or of analytical tools) by working in the Banach space $X$ considered in Sect. 7.1.1.

## 7.2 Sub- and Super-Solutions and Increasing Operators

A separate discussion is in order when we want to use the topics studied in Sect. 2.2 dealing with increasing operators.

In this case, by definiteness we fix $\lambda = 1$ in (7.1), i.e., let us consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(x, u), & \text{in } \Omega \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7.5}$$

where $f \in C^{0,\nu}(\overline{\Omega} \times \mathbb{R})$, $0 < \nu < 1$, and there exists $m > 0$ such that

$(f0)$ For every fixed $x \in \overline{\Omega}$, the function $f_m(x, u) := f(x, u) + mu$ is increasing with respect to $u$.

Problem (7.5) is equivalent to

$$\begin{cases} -\Delta u + mu = f_m(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.6}$$

According to the previous arguments, (7.6) can be translated into the operator equation $u = T(u)$, with $u \in X = C^{0,\nu}(\overline{\Omega})$, $0 \le \nu < 1$, and $T = K_m \circ f_m$, where $K_m u = z$ iff $-\Delta z + mz = u$, $z|_{\partial\Omega} = 0$. Let us remark that $T$ is compact since $K_m$ is. In the space $X$ we consider the natural ordering: $v \le w$ iff $v(x) \le w(x)$ for all $x \in \overline{\Omega}$.

We claim that $T$ is an increasing operator. Let $u \le v$; then $z = T(u) = K_m(f_m(u))$ (respectively, $w = T(v) = K_m(f_m(v))$ is a solution of the equation $-\Delta z + mz = f_m(x,u)$ (resp. $-\Delta w + mw = f_m(x,v)$), satisfying $z|_{\partial\Omega} = w|_{\partial\Omega} = 0$. By $(f0)$, $u \le v$ implies that $f_m(x,u) \le f_m(x,v)$ and hence, by the maximum principle (see Theorem 1.3.14), it follows that $z \le w$ in $\Omega$, proving the claim.

A function $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta v \le f(x,v), & \text{in } \Omega \\ v \le 0, & \text{on } \partial\Omega, \end{cases}$$

is called a sub-solution of (7.5). Similarly, a super-solution $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is defined by requiring

$$\begin{cases} -\Delta w \ge f(x,w), & \text{in } \Omega \\ w \ge 0, & \text{on } \partial\Omega. \end{cases}$$

Thus a sub-solution, resp. super-solution, is a $v \in X$, resp. $w \in X$, such that $v \le T(v)$, resp. $w \ge T(w)$.

After these preliminaries, a straight application of Theorem 2.2.2 yields the following.

**Theorem 7.2.1** *Let $f \in C^{0,\nu}(\overline{\Omega} \times \mathbb{R})$, $0 < \nu < 1$ satisfy $(f0)$ and suppose that $v$, resp. $w$, is a sub-solution, resp. super-solution, of* (7.5) *such that $v \le w$. Then* (7.5) *has a solution $u$ such that $v \le u \le w$. Moreover,* (7.5) *has a* minimal *solution $u_1$ and a* maximal *solution $u_2$, in the sense that any other solution $u$ of* (7.5)*, such that $v \le u \le w$, satisfies $u_1 \le u \le u_2$.*

*Remark 7.2.2* The proof of the previous theorem is also obtained by imposing only a more general hypothesis than $(f0)$. Indeed, it instead suffices to assume that the following condition holds:

$(\widetilde{f}0)$ For every fixed $x \in \overline{\Omega}$, the function $f_m(x,u) := f(x,u) + mu$ is increasing with respect to $u \in [\min_{\overline{\Omega}} v, \max_{\overline{\Omega}} w]$.

Notice that every locally Lipschitzian function $f$ satisfies the hypothesis $(\widetilde{f}0)$.

As a trivial application of Theorem 7.2.1 let us show that (7.5) possesses a solution provided $f \in C^{0,\nu}(\overline{\Omega} \times \mathbb{R})$, $0 < \nu < 1$ and there exist $a < b$ such that $f(x,a) \ge$

$0 \geq f(x,b)$ for all $x \in \Omega$. It suffices to remark that $v \equiv a$ is a sub-solution, and $w \equiv b$ is a super-solution of (7.5), because

$$0 = -\Delta v \leq f(x,a), \quad 0 = -\Delta w \geq f(x,b).$$

Moreover $v < w$ and Theorem 7.2.1 yields a solution $u$ of (7.5).

It will be useful to know different proofs of Theorem 7.2.1. Using topological or variational methods we will find additional information about the solution. This will provide the existence of multiple solutions. We begin by computing the degree of the operator $I - T$.

**Lemma 7.2.3** *In addition to the hypotheses of Theorem 7.2.1, assume that* $f \in C^1(\overline{\Omega} \times \mathbb{R})$ *and that* $v, w \in C_0^1(\overline{\Omega}) \cap C^2(\Omega)$ *are not solutions,* $v < w$ *in* $\Omega$ *and* $\frac{\partial w}{\partial n} < \frac{\partial v}{\partial n}$ *on* $\partial\Omega$. *For* $R > 0$ *we set*

$$U(R) = \{u \in C_0^1(\overline{\Omega}) : \; v < u < w \; in \; \Omega, \; \frac{\partial w}{\partial n} < \frac{\partial u}{\partial n} < \frac{\partial v}{\partial n} \; on \; \partial\Omega\} \cap B_R(0),$$

*where* $B_R(0)$ *denotes the ball centered at zero and with radius* $R$ *in the space* $C_0^1(\overline{\Omega})$ *of the functions of class* $C^1$ *which vanish on* $\partial\Omega$. *Then there is* $R > 0$ *such that* $deg(I - T, U(R), 0) = 1$.

*Proof* Observe that the set $U(R)$ is open, bounded and convex in $C_0^1(\overline{\Omega})$. We consider the truncated problem

$$\begin{cases} -\Delta u + mu = \widetilde{f}_m(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where

$$m = \max\left\{\left|\frac{\partial f}{\partial s}(x,s)\right| \; : \; x \in \Omega, \; s \in \left[\min_{\overline{\Omega}} v, \max_{\overline{\Omega}} w\right]\right\}$$

and

$$\widetilde{f}_m(x,s) = \begin{cases} f(x,v(x)) + mv(x), & \text{if } s < v(x) \\ f(x,s) + ms, & \text{if } v(x) \leq s \leq w(x) \\ f(x,w(x)) + mw(x), & \text{if } w(x) < s. \end{cases}$$

Following the notation used in Theorem 7.2.1, we set $\widetilde{T} = K_m \circ \widetilde{f}_m$, which is a compact operator (by the compactness of $K_m$). In addition, the boundedness of $\widetilde{f}_m$ and the $L^p$-estimates imply that $T(C_0^1(\overline{\Omega}))$ is bounded in $C_0^1(\overline{\Omega})$. Let $R$ be a positive number such that $T(C_0^1(\overline{\Omega})) \subset B(R)$. Using the strong comparison principle and the Hopf lemma (see [58, Lemma 3.4]) for the operator $-\Delta + mI$ it is easily seen that $v < \widetilde{T}(u) < w$ in $\Omega$ and $\frac{\partial w}{\partial n} < \frac{\partial \widetilde{T}(u)}{\partial n} < \frac{\partial v}{\partial n}$ on $\partial\Omega$ for every $u \in C_0^1(\overline{\Omega})$. In other words, we have seen that $\widetilde{T}(C_0^1(\overline{\Omega})) \subset U(R)$. Fixing $z \in U(R)$, let us consider the homotopy

$$H(\lambda, u) = \lambda \widetilde{T}(u) + (1 - \lambda)z.$$

The convexity of $U(R)$ implies that $\lambda \widetilde{T}(u) + (1 - \lambda)z \in U(R)$ for every $\lambda \in [0, 1]$ and $u \in C_0^1(\overline{\Omega})$. Hence,

$$u \neq H(\lambda, u), \quad \forall u \in \partial U(R), \quad \forall \lambda \in [0, 1]$$

and $\deg(H(\lambda, \cdot), U(R), 0)$ is well defined. The invariance by homotopy gives

$$\deg(I - \widetilde{T}, U(R), 0) = \deg(H(1, \cdot), U(R), 0) = \deg(H(0, \cdot), U(R), 0) = 1$$

since $H(0, \cdot)$ is a constant map. □

Now, we show that in general the solution obtained between sub- and super-solution is a local minimum of the corresponding Euler functional:

$$\mathcal{J}(u) = \frac{1}{2} \int |\nabla u|^2 - \int F(x, u), \; u \in H_0^1(\Omega),$$

where, as usual, $F(x, s) = \int_0^s f(x, t) dt$. In order to do his, we have to prove the following result due to Brezis and Nirenberg [38].

**Theorem 7.2.4** *Assume that*

$$|f(x, u)| \leq c_1 + c_2 |u|^p, \; a.e. \; x \in \Omega, \quad \forall u \in \mathbb{R}$$

*with* $p < 2^* - 1$. *If, for some* $r > 0$, $u_0 \in H_0^1(\Omega)$ *satisfies*

$$\mathcal{J}(u_0) \leq \mathcal{J}(u_0 + v), \quad \forall v \in C_0^1(\overline{\Omega}) \textit{ with } \|v\|_{C^1} \leq r, \tag{7.7}$$

*then there exists* $\varepsilon > 0$ *such that*

$$\mathcal{J}(u_0) \leq \mathcal{J}(u_0 + v), \quad \forall v \in H_0^1(\Omega) \textit{ with } \|v\| \leq \varepsilon.$$

*Remark 7.2.5* 1. In other words, if $u_0 \in H_0^1(\Omega)$ is a local minimizer of $\mathcal{J}$ in the $C^1$-topology, then $u_0$ is also a local minimizer of $\mathcal{J}$ in the $H_0^1$-topology.

2. As can be seen in the original paper by Brezis and Nirenberg, the theorem is also true for $p = 2^* - 1$.

*Proof* By $L^p$-theory it is possible to show that $u_0 \in C^1(\overline{\Omega})$ and thus we may assume without loss of generality that $u_0 = 0$. We argue by contradiction assuming that for every $\varepsilon > 0$ there exists $v_\varepsilon \in H_0^1(\Omega)$ such that $\|v_\varepsilon\| \leq \varepsilon$ and

$$\mathcal{J}(v_\varepsilon) = \min_{\|v\| \leq \varepsilon} \mathcal{J}(v) < \mathcal{J}(0).$$

(Observe that the existence of minimizer $v_\varepsilon$ is a consequence of the weak lower semicontinuity of $\mathcal{J}$). By the Lagrange multiplier theorem, there exists a Lagrange multiplier $\mu_\varepsilon \geq 0$ such that the minimizer $v_\varepsilon$ satisfies $-\Delta v_\varepsilon - f(x, v_\varepsilon) = -\mu_\varepsilon \Delta v_\varepsilon$, i.e.,

$$-\Delta v_\varepsilon = \frac{f(x, v_\varepsilon)}{1 + \mu_\varepsilon}.$$

Since $p < 2^* - 1$, the Agmong–Douglis–Nirenberg regularity result implies that the norm $\|v_\varepsilon\|_{C^1}$ may be estimated by the norm $\|v_\varepsilon\|$ which is smaller than or equal to $\varepsilon$. Therefore, $\|v_\varepsilon\|_{C^1}$ converges to zero as $\varepsilon$ goes to zero and we may choose $\varepsilon$ such that $\|v_\varepsilon\|_{C^1} \leq r$ and then, by (7.7), $\mathcal{J}(0) \leq \mathcal{J}(v_\varepsilon)$, contradicting the definition of $v_\varepsilon$. □

**Lemma 7.2.6** *If, in addition to the assumptions of Theorem 7.2.1, $v$ and $w$ are not solutions of (7.5), then there is solution $u$ of (7.5) with $v < u < w$ in $\Omega$ and which is a local minimizer of $\mathcal{J}$ in $H_0^1(\Omega)$.*

*Proof* For the convenience of the reader, here we prove the lemma in the case that $v, w \in C_0^1(\overline{\Omega})$, which allows us to use the Hopf lemma (see [58, Lemma 3.4]). We refer to [38] for the general case (even if $v$ and $w$ are only continuous (not $C^2$) sub- and super- solutions in the sense of distributions). We consider now the truncated problem

$$\begin{cases} -\Delta u = \widetilde{f}(x, u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where

$$\widetilde{f}(x, s) = \begin{cases} f(x, v(x)), & \text{if } s < v(x) \\ f(x, s), & \text{if } v(x) \leq s \leq w(x) \\ f(x, w(x)), & \text{if } w(x) < s. \end{cases}$$

Since $\widetilde{f}$ is bounded, a solution $u_0 \in H_0^1(\Omega) \cap C_0^1(\overline{\Omega})$ of the preceding Dirichlet problem can be obtained by minimization of the coercive functional

$$\widetilde{\mathcal{J}}(u) = \frac{1}{2} \int |\nabla u|^2 - \int \widetilde{F}(x, u), \; u \in H_0^1(\Omega),$$

with $\widetilde{F}(x, s) = \int_0^s \widetilde{f}(x, t)\, dt$ (see Example 5.2.1). Using that $v$ is a sub-solution,

$$-\Delta(v - u_0) \leq f(x, v) - \widetilde{f}(x, u_0) \tag{7.8}$$

and consequently, if the set $A = \{x \in \Omega : u_0(x) < v(x)\}$ were not empty, then $-\Delta(v - u_0) \leq 0$ in $A$ and $v - u_0 \leq 0$ on $\partial A$. By the maximum principle we deduce that $v \leq u_0$ in $A$, contradicting the definition of $A$ and proving that $A = \emptyset$, i.e., $v \leq u_0$.

Using again (7.8) and hypothesis $(f0)$ we also obtain $-\Delta(v - u_0) + k(v - u_0) \leq [f(x, v) + k(v - u_0)] - [\widetilde{f}(x, u_0) + k(v - u_0)] \leq 0$. Taking into account that $v$ is not a solution, $v - u_0 \not\equiv 0$, the strong maximum principle and Hopf lemma show that $u_0 - v$ belongs to the interior of the cone of positive functions in $C_0^1(\overline{\Omega})$.

A similar argument using the super-solution $w$ (instead of $v$) implies that $w - u_0$ is also in the interior of this cone. Therefore, $u_0$ is in the interior of the set $\{u \in C_0^1(\overline{\Omega}) : v \le u \le w \text{ in } \Omega\}$, i.e., there exists $\varepsilon > 0$ such that

$$\left.\begin{array}{c} u \in C_0^1(\overline{\Omega}) \\ \|u - u_0\|_{C^1} \le \varepsilon \end{array}\right\} \Longrightarrow v \le u \le w \text{ in } \Omega.$$

To conclude the proof, it suffices to observe that $\mathcal{J}(u) - \widetilde{\mathcal{J}}(u)$ is constant in the ball $\|u - u_0\|_{C^1} \le \varepsilon$ (since $\widetilde{F}(x,u) - F(x,u)$ depends only on $x$ for $u \in [v(x), w(x)]$) and to use that $u_0$ is a global minimizer of $\widetilde{\mathcal{J}}$. This implies that $u_0$ is a local minimizer of $\mathcal{J}$ in the $C^1$-topology and, by Theorem 7.2.4, in the $H_0^1(\Omega)$-topology. □

The following example shows that, in general, the existence of a sub-solution $v$ and a super-solution $w$, without assuming that they are ordered $v \le w$, does not imply the existence of a solution. Indeed, let $\lambda_1 < \lambda_2$ denote the first two eigenvalues of the linear Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and let $\varphi_1 > 0$, and $\varphi_2 \not\equiv 0$ denote two eigenfunctions corresponding to $\lambda_1, \lambda_2$, resp. Fixing $0 < \alpha < \beta$, choose a smooth function $h$ such that (i) $\alpha(\lambda_2 - \lambda_1)\varphi_1 < h < \beta(\lambda_2 - \lambda_1)\varphi_1$ and (ii) $\int h\varphi_2 \neq 0$. Using $\varphi_2$ as a test function, the latter condition implies that the problem

$$\begin{cases} -\Delta u = \lambda_2 u - h(x), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{7.9}$$

has no solution. On the other hand,

$$v = \beta\varphi_1, \quad w = \alpha\varphi_1,$$

are, respectively, a sub-solution and a super-solution of (7.9). Actually, $v(x) = w(x) = 0$ for $x \in \partial\Omega$ and, using (i), we find that

$$-\Delta v = \beta\lambda_1\varphi_1 < \lambda_2\beta\varphi_1 - h = \lambda_2 v - h,$$

as well as

$$-\Delta w = \alpha\lambda_1\varphi_1 < \lambda_2\alpha\varphi_1 - h = \lambda_2 w - h.$$

Despite the above example, we see now a result in which the existence of a sub- and a super-solution is sufficient to find a solution of a boundary value problem. Specifically, we follow [3] to prove the following theorem.

**Theorem 7.2.7** *Let $g$ be a bounded continuous function. If there exist a sub-solution $v$ and a super-solution $w$ of the problem*

$$\begin{cases} -\Delta u = \lambda_1 u + g(u), & \textit{in } \Omega \\ u = 0, & \textit{on } \partial\Omega, \end{cases}$$

*then there exists a solution of it.*

*Proof* Consider $X = C(\overline{\Omega})$ and take a positive eigenfunction $\varphi$ associated to the first eigenvalue $\lambda_1$. Then Ker $(-\Delta - \lambda_1 I) = \mathbb{R}\varphi$. We split the space $X = \mathbb{R}\varphi \oplus W$, i.e. $u = t\varphi + w$, where $t \in \mathbb{R}$ and $w \in W$. By the Lyapunov–Schmidt reduction we see that our problem is equivalent to the system

$$w = AQg(t\varphi + w)$$
$$\int g(t\varphi + w)\varphi = 0,$$

where $A$ is the inverse of the operator $-\Delta - \lambda_1 I$ in $W$ (since zero is not an eigenvalue of it).

We study now the solution set $\Sigma = \{(t, w) \in \mathbb{R} \times W \,:\, w = AQb(t\varphi + w)\}$ of the first equation in the system. By the boundedness of $g$, there exists $R > 0$ such that

$$\|AQg(t\varphi + w)\| < r, \quad \forall t \in \mathbb{R}, \quad \forall w \in W$$

and hence $\Sigma \subset \mathbb{R} \times B_r$, where $B_r$ denotes the open ball in $W$ of radius $r$ and centered at zero. This implies also that $\Phi_\lambda(w) = w - \lambda AQg(t\varphi + w)$, for $\lambda \in [0, 1]$ and $w \in W$, defines a homotopy and thus

$$\deg(\Phi_\lambda, B_r, 0) = \deg(\Phi_0, B_r, 0) = 1.$$

Given $\alpha > 0$, by using Theorem 4.3.4 we obtain the existence of a connected set $\Sigma_\alpha \subset \Sigma$ such that $\Sigma_\alpha$ crosses $\{-\alpha\} \times W$ as well as $\{\alpha\} \times W$. Taking into account that $\Sigma_\alpha$ is connected and the continuity on $\mathbb{R} \times W$ of the function $\gamma(t, w) = \int g(t\varphi + w)\varphi$ we deduce that $\gamma(\Sigma_\alpha)$ is an interval. Three cases may occur: $0 \in \gamma(\Sigma_\alpha)$, $\gamma(\Sigma_\alpha) \subset (0, +\infty)$ or $\gamma(\Sigma_\alpha) \subset (-\infty, 0)$.

In the first case, we have already solved the system and so our problem.

With respect to the second case, i.e., $\gamma(\Sigma_\alpha) \subset (0, +\infty)$, we deduce that for every pair $(t, z) \in \mathbb{R} \times Z$ the function $u = t\varphi + w$ satisfies

$$-\Delta u = -t\Delta\varphi - \Delta w = \lambda_1 t\varphi + \lambda_1 w + g(t\varphi + w) - \gamma(t, w)\varphi < \lambda_1 u + g(u), \text{ in } \Omega,$$

namely, it is a sub-solution of our problem. Clearly it is possible to take a very negative number $t$ in order to have $u$ be smaller than or equal to the super-solution $w$ given by hypothesis. Consequently, Theorem 7.2.1 applies and we also obtain a solution of our problem in this case.

The proof in the third case is similar: it suffices to observe now that $u = t\varphi + w$ is a super-solution. □

# Chapter 8
# Problems with A Priori Bounds

In this chapter we discuss problems in which one can obtain a priori bounds for the solutions. Roughly, this happens if the nonlinearity is sublinear at infinity. It will be shown that, according to the properties of the nonlinearity, we can use the global inversion theorem (to get existence and uniqueness) or topological degree or else critical point theory.

## 8.1 An Elementary Nonexistence Result

Consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(u) + h(x), & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{8.1}$$

In this case, the existence of a priori bounds for (8.1) is strongly related to the fact that the limits at infinity of $f(u)/u$ do not intersect the spectrum of the Laplace operator, in a sense made precise in the sequel.

To put more in evidence the necessity of interaction with spectrum, observe that if (8.1) has one solution, using a positive eigenfunction $\varphi_1$ associated to the first eigenvalue $\lambda_1$ of the Laplace operator as a test function, we conclude that

$$\int [f(u) + h(x) - \lambda_1 u]\, \varphi_1 = 0.$$

Consequently we deduce the following trivial nonexistence result.

**Proposition 8.1.1** *Assume that $\Omega$ is an open subset in $\mathbb{R}^N$, $f$ is continuous and $h \in L^2(\Omega)$. If either*

- *$f(u) + h(x) < \lambda_1 u$ for every $u \in \mathbb{R}$ and a.e. $x \in \Omega$, or*
- *$f(u) + h(x) > \lambda_1 u$ for every $u \in \mathbb{R}$ and a.e. $x \in \Omega$,*

*then problem (8.1) has no weak solution.* □

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82,
DOI 10.1007/978-0-8176-8114-2_8, 

## 8.2 Existence of A Priori Bounds

Let $f \in C^1(\mathbb{R}, \mathbb{R})$ and assume that the following limits exist:

$$\lim_{u \to -\infty} \frac{f(u)}{u} := \gamma_- \in \mathbb{R}, \qquad \lim_{u \to +\infty} \frac{f(u)}{u} := \gamma_+ \in \mathbb{R}. \tag{8.2}$$

We denote by $\Gamma$ the closed interval of extrema $\gamma_-$ and $\gamma_+$.

For fixed $0 < \nu < 1$, let $X = \{u \in C^{2,\nu}(\overline{\Omega}) : u(x) = 0 \text{ on } \partial\Omega\}$, $Y = C^{0,\nu}(\overline{\Omega})$ and $F(u) = -\Delta u - f(u)$. For any $h \in Y$, the classical solutions of (8.1) are $u \in X$ such that $F(u) = h$.

**Proposition 8.2.1** *Suppose that* (8.2) *holds and that the interval* $\Gamma$ *does not contain any eigenvalue* $\lambda_k$. *Let* $u_n \in X$ *and set* $h_n = F(u_n)$. *Then* $u_n$ *is bounded in* $X$ *provided that* $h_n \in Y$ *is.*

*Proof* We start by proving that $\|u_n\|_Y$ is bounded. Otherwise, up to a subsequence, we can assume that $\|u_n\|_Y$ converges to infinity. Setting $z_n = u_n \|u_n\|_Y^{-1}$ and using (8.2) to write

$$f(s) = \gamma_+ s^+ + \gamma_- s^- + g(s), \qquad \text{with} \qquad \lim_{|s| \to +\infty} g(s)s^{-1} = 0,$$

we immediately check that $z_n$ satisfies

$$-\Delta z_n = \gamma_+ z_n^+ + \gamma_- z_n^- + \frac{g(u_n)}{\|u_n\|_Y} + \frac{h_n}{\|u_n\|_Y}. \tag{8.3}$$

The right-hand side is bounded in $C(\overline{\Omega})$ and hence by Schauder estimates, $z_n$ is bounded in $C^{1,\nu}(\overline{\Omega})$. Then, up to a subsequence, $z_n$ converges strongly to some $z$ in $C^1(\overline{\Omega})$ with $\|z\|_Y = 1$. Using a test function $\phi$ we get

$$\int \nabla z_n \cdot \nabla \phi = \int \gamma_+ z_n^+ \phi + \int \gamma_- z_n^- \phi + \int g(u_n) z_n \phi + \int \frac{h_n}{\|u_n\|_Y} \phi.$$

By using the dominated convergence theorem we have

$$\lim_{n \to +\infty} \int \frac{g(u_n(x))}{\|u_n\|_Y} \phi = 0,$$

and hence

$$\int \nabla z \cdot \nabla \phi = \int \left[\gamma_+ z^+ + \gamma_- z^-\right] \phi.$$

This means that $z \neq 0$ satisfies

$$\begin{cases} -\Delta z = a(x) z, & \text{in } \Omega, \\ z = 0, & \text{on } \partial\Omega, \end{cases} \tag{8.4}$$

where

$$a(x) := \begin{cases} \gamma_+, & \text{if } z(x) \geq 0; \\ \gamma_-, & \text{if } z(x) < 0. \end{cases} \tag{8.5}$$

Then $\lambda_{\overline{k}}[a] = 1$ for some integer $\overline{k} \geq 1$. We assume now that $\gamma_+ \leq \gamma_-$ (the verification for the reversed case $\gamma_- \leq \gamma_+$ is left to the reader). Observe that if $z$ does not change sign then either $-\Delta z = \gamma_+ z^+$ or $-\Delta z = \gamma_- z^-$. Both cases are not possible because $\Gamma = [\gamma_+, \gamma_-]$ does not contain any eigenvalue $\lambda_k$. Hence $u$ changes sign and the following holds:

$$|\{x \in \Omega : a(x) > \gamma_+\}| > 0 \text{ and } |\{x \in \Omega : a(x) < \gamma_-\}| > 0.$$

Using the comparison property of eigenvalues (see Proposition 1.3.11-i)), we obtain

$$\frac{\lambda_{\overline{k}}}{\gamma_+} = \lambda_{\overline{k}}[\gamma_+] > \lambda_{\overline{k}}[a] = 1 > \lambda_{\overline{k}}[\gamma_-] = \frac{\lambda_{\overline{k}}}{\gamma_-},$$

i.e.,

$$\gamma_+ < \lambda_{\overline{k}} < \gamma_- ,$$

a contradiction because $\Gamma = [\gamma_+, \gamma_-]$ does not contain any eigenvalue $\lambda_k$. Therefore, $\|u_n\|_Y$ is bounded and Schauder estimates (see Theorem 1.2.11) imply that $\|u_n\|_X$ is also bounded. □

## 8.3 Existence of Solutions

In this section we will see how the different abstract techniques developed in the previous chapters can be used in conjunction with the a priori bounds of the previous section to prove existence of solutions of the problem (8.1). We will keep the notation introduced before.

### *8.3.1 Using the Global Inversion Theorem*

Our first result deals with a case in which the global inversion Theorem 3.4.5 applies. We start by proving the following lemma.

**Lemma 8.3.1** *Suppose that $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies (8.2). If the interval $\Gamma$ does not contain any eigenvalue $\lambda_k$, then $F$ is proper.*

*Proof* Let $h_n \in Y$, $h_n \to h \in Y$ and let $u_n \in X$ be such that $F(u_n) = h_n$, namely

$$\begin{cases} -\Delta u_n = f(u_n) + h_n(x), & \text{in } \Omega, \\ \quad u_n = 0, & \text{on } \partial\Omega. \end{cases}$$

By Proposition 8.2.1, $u_n$ is bounded in $X$. In particular, $f(u_n) + h_n$ is also bounded in $Y$. By the compactness of $K : Y \longrightarrow Y$ we deduce that, up to a subsequence, $u_n$ converges in $Y$ to some $u \in X$ satisfying $-\Delta u = f(u) + h(x)$, in $\Omega$. Finally, using that $-\Delta(u_n - u) = f(u_n) - f(u) + h_n(x) - h(x)$ and Schauder estimates (see Theorem 1.2.11), we readily conclude that $u_n$ converges in $X$ to $u$. □

**Theorem 8.3.2** *Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be such that*

1. *condition (8.2) is satisfied and the interval $\Gamma$ does not contain any eigenvalue $\lambda_k$;*
2. *for all $u \in \mathbb{R}$, either $f'(u) < \lambda_1$, or $\lambda_k < f'(u) < \lambda_{k+1}$ for some $k \geq 1$.*

*Then (8.1) has a unique solution for all $h \in Y$.*

*Proof* Let us show that $F$ has no singular points. One has

$$dF(u)[v] = \Delta v + f'(u)v.$$

Using assumption 2 and the comparison property of the eigenvalues, it readily follows that $\operatorname{Ker} dF(u) = \{0\}$ and this implies that the singular set is empty. By hypothesis 1 (see Lemma 8.3.1), $F$ is proper and the global inversion theorem applies, proving the result. □

### *8.3.2 Using Degree Theory*

We apply the Schauder fixed point Theorem 4.2.6 to prove the existence part of Theorem 8.3.2.

**Theorem 8.3.3** *Suppose that $f \in C(\mathbb{R}, \mathbb{R})$ satisfies (8.2) and the interval $\Gamma$ does not contain any eigenvalue $\lambda_k$. Then (8.1) has at least one solution.*

*Proof* We will carry out the proof in the case $\lambda_k < \gamma_-, \gamma_+ < \lambda_{k+1}$. If $\gamma_-, \gamma_+ < \lambda_1$ the arguments require trivial changes. We let $\gamma = \frac{\lambda_k + \lambda_{k+1}}{2}$ and denote by $K_\gamma : L^2(\Omega) \longrightarrow L^2(\Omega)$ the inverse of the operator $-\Delta - \gamma I$ (which is well defined because $\gamma$ is not an eigenvalue of the Laplacian operator). We write (8.1) as

$$u = K_\gamma[N(u)], \quad u \in L^2(\Omega),$$

where $N : L^2(\Omega) \longrightarrow L^2(\Omega)$ is defined by

$$Nu(x) = f(u(x)) + h(x) - \gamma u(x), \quad \forall x \in \Omega.$$

Hence, we just have to prove the existence of a fixed point of $T = K_\gamma \circ N$. This is done by using the Schauder fixed point theorem. In order to do this, observe that from the hypotheses on $f$, the continuous function $g$ defined in $\overline{\Omega} \times \mathbb{R}$ by

$$g(x, s) = f(x, s) - \gamma_+ s^+ - \gamma_- s^-, \quad \forall s \in \mathbb{R}$$

(with $s^+ = \max\{s, 0\}$ and $s^- = \min\{s, 0\}$) satisfies $\lim_{|s| \to +\infty} g(x, s)/s = 0$ (i.e., it is sublinear). Hence, there exists $\overline{\mu} \in (0, (\lambda_{k+1} - \lambda_k)/2)$ such that

$$|f(x, s) - \gamma s| \leq \overline{\mu}|s|, \quad \forall s \in \mathbb{R}$$

and thus

$$\begin{aligned}\|Tu\|_2 &\leq \|K_\gamma\|\|N(u)\|_2\\ &\leq \frac{2}{\lambda_{k+1}-\lambda_k}\left[\|f(x,u)-\gamma u\|_2+\|h\|_2\right]\\ &\leq \frac{2}{\lambda_{k+1}-\lambda_k}\left[\overline{\mu}\|u\|_2+\|h\|_2\right]\\ &\leq \mu\|u\|_2+C,\end{aligned}$$

where $\mu = \frac{2\overline{\mu}}{\lambda_{k+1}-\lambda_k} \in (0,1)$. So choosing $r > 0$ such that $\mu r + C < r$, we deduce that $\|Tu\|_{L^2} \leq r$, provided $\|u\|_{L^2} \leq r$, i.e., $T(\overline{B}(0,r)) \subset \overline{B}(0,r)$. Finally the compactness of $K_\gamma$ and the continuity of the Nemitski operator $N$ (see Theorem 1.2.1) imply the compactness of $T$, and we can so use the Schauder Theorem 4.2.6 to conclude the proof. □

*Remark 8.3.4* From the proof of Schauder Theorem 4.2.6 by using the degree, we deduce that there exists $r > 0$ such that $\deg(T, B(0,r), 0) = 1$.

*Remark 8.3.5* If, in addition, item 2 of Theorem 8.3.2 is satisfied, then it is possible to prove the uniqueness of the solution as well. Indeed, the meaning of this condition is that the operator $T$ is a contraction and thus the Banach contraction principle can be applied to obtain (existence and) uniqueness.

### *8.3.3 Using Critical Point Theory*

According to the discussion of Sect. 7.1.2, solutions of (8.1) could also be found by looking for critical points of the Euler functional

$$\mathcal{J}(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(u) - \int h(x)u, \quad u \in E = H_0^1(\Omega),$$

where $F(u) = \int_0^u f(s)ds$. Let us remark that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|F(u)| \leq C_\varepsilon + \frac{1}{2}\left(\max\{\gamma_-, \gamma_+\} + \varepsilon\right)|u|^2. \tag{8.6}$$

In particular, $F(u) \in L^1(\Omega)$ for all $u \in E$ and $\mathcal{J} \in C^1(E, \mathbb{R})$. Moreover, setting

$$\mathcal{F}(u) = \int F(u),$$

one has the following.

**Lemma 8.3.6** *$\mathcal{F}$ is weakly continuous.*

*Proof* Let $u_n \rightharpoonup u$ weakly in $E$. By the Rellich–Kondrachov Theorem A.4.9, $u_n \to u$ strongly in $L^2$, up to a subsequence. From (8.6) and the dominated convergence theorem, we deduce that $\mathcal{F}(u_n) \to \mathcal{F}(u)$. □

In order to prove Theorem 8.3.3 by applying the results on critical point theory to the functional $J$, we state the following lemma on the Palais–Smale condition.

**Lemma 8.3.7** *If $\Gamma$ does not contain any eigenvalue $\lambda_k$, then $\mathcal{J}$ satisfies the Palais–Smale condition.*

*Remark 8.3.8* The reader should observe the similarity between the proof of this lemma and the corresponding proof of Proposition 8.2.1.

*Proof* Let $\{u_n\} \subset H_0^1(\Omega)$ be a sequence such that $\{\mathcal{J}(u_n)\}$ is bounded and $\{\mathcal{J}'(u_n)\}$ tends to zero in $H_0^1(\Omega)$. To prove that $\{u_n\}$ has a convergent subsequence, it suffices to show that it is bounded in $H_0^1(\Omega)$ (see Lemma 7.1.1). Assume, by contradiction, that $\|u_n\| \to +\infty$. Using that

$$\lim_{n\to+\infty} \frac{\mathcal{J}'(u_n)(\varphi)}{\|u_n\|} = 0$$

and taking $z_n \equiv u_n/\|u_n\|$, we obtain

$$\lim_{n\to+\infty} \int \nabla z_n \cdot \nabla\varphi - \int \frac{f(u_n)}{\|u_n\|}\varphi - \int \frac{h\varphi}{\|u_n\|} = 0,$$

for every $\varphi \in H_0^1(\Omega)$. Passing to a subsequence if necessary, we may assume without loss of generality that $z_n \rightharpoonup z$ in $H_0^1(\Omega)$, $z_n \to z$ in $L^2(\Omega)$, $z_n(x) \to z(x)$ a.e. $x \in \Omega$. Thus, by the Lebesgue dominated convergence theorem we yield

$$\lim_{n\to+\infty} \int \frac{f(u_n)}{\|u_n\|}\varphi = \int \left(\gamma_+ z^+ + \gamma_- z^-\right)\varphi$$

and hence

$$\int \nabla z \cdot \nabla\varphi = \int \left(\gamma_+ z^+ + \gamma_- z^-\right)\varphi,$$

i.e., $v$ is a solution of the problem (8.4). This implies that $z = 0$ and this is a contradiction because we deduce that

$$0 = \lim_{n\to+\infty} (J'(u_n) \mid z_n) = 1 - \lim_{n\to+\infty} \int f(x, u_n)z_n - \int hz_n = 1.$$

Therefore, $\{u_n\}$ is bounded and the Palais–Smale condition has been verified. □

*Variational proof of Theorem 8.3.3* By Lemma 8.3.7, it suffices to study the geometry of $\mathcal{J}$. It is convenient to consider three cases:

Case 1. $\max\{\gamma_-, \gamma_+\} < \lambda_1$
Case 2. $\lambda_1 < \gamma_-, \gamma_+ < \lambda_2$
Case 3. $\lambda_k < \gamma_-, \gamma_+ < \lambda_k + 1$, with $k \geq 2$.

*Case 1* We begin proving that if $\max\{\gamma_-, \gamma_+\} < \lambda_1$, then $\mathcal{J}$ has a minimum on $E$ which gives rise to a solution (8.9). Indeed, if we fix $\overline{\gamma} \in (\max\{\gamma_-, \gamma_+\}, \lambda_1)$, from (8.6) one has

$$\mathcal{J}(u) \geq \tfrac{1}{2}\|u\|^2 - c_1 - \tfrac{1}{2}\,\overline{\gamma}\int |u|^2 - \|h\|_2\|u\|_{L^2}.$$

Using the Poincaré inequality (Corollary 1.3.9) we deduce

$$\mathcal{J}(u) \geq \tfrac{1}{2}\left(1 - \frac{\overline{\gamma}}{\lambda_1}\right)\|u\|^2 - c_1 - c_2\|u\|.$$

Since $\overline{\gamma} < \lambda_1$, it follows that $\mathcal{J}$ is coercive and then, by Theorem 5.4.1-2, $\mathcal{J}$ has a minimum on $E$. (It is also possible to show that $\mathcal{J}$ is w.l.s.c. and therefore to apply Corollary 1.2.5 instead of Theorem 5.4.1-2, to prove the existence of a minimum of $\mathcal{J}$ on $E$.)

*Case 2* We prove that if $\lambda_1 < \gamma_-, \gamma_+ < \lambda_2$ the functional $\mathcal{J}$ has a mountain pass critical point. Indeed, choosing $\lambda_1 < \mu < \gamma_-, \gamma_+ < \overline{\mu} < \lambda_2$, it is easy to verify that

$$\frac{1}{2}\mu u^2 - C_1 \leq F(u) \leq \frac{1}{2}\overline{\mu}u^2 + C_2, \quad \forall u \in \mathbb{R},$$

and thus

$$\mathcal{J}(u) \leq \frac{1}{2}\|u\|^2 - \frac{\mu}{2}\|u\|_2^2 + C_1|\Omega| + \|h\|_{L^2}\|u\|_2, \tag{8.7}$$

$$\mathcal{J}(u) \geq \frac{1}{2}\|u\|^2 - \frac{\overline{\mu}}{2}\|u\|_{L^2}^2 - C_2|\Omega| - \|h\|_{L^2}\|u\|_{L^2}. \tag{8.8}$$

From (8.7) it follows that

$$\lim_{|t|\to+\infty} \mathcal{J}(t\varphi_1) = -\infty.$$

From (8.8) it follows that

$$\inf_{\langle\varphi_1\rangle^\perp} \mathcal{J} > -\infty.$$

Moreover, the Palais–Smale condition holds (see Lemma 8.3.7). Then we can use the variant of the mountain pass theorem given in Theorem 5.3.8 to infer that $\mathcal{J}$ has a critical point.

*Case 3* We show that if $\lambda_k < \gamma_-, \gamma_+ < \lambda_{k+1}$, with $k \geq 2$, then the saddle point theorem applies.

Indeed, choosing $\lambda_k < \mu < \gamma_-, \gamma_+ < \overline{\mu} < \lambda_{k+1}$, and repeating the arguments carried out in case 2, it is easy to verify that $\mathcal{J}$ satisfies (8.7) and (8.8).

Splitting $H_0^1(\Omega)$ into $H_0^1(\Omega) = V \oplus V^\perp$ with $V = \langle \varphi_1, \varphi_2, \dots, \varphi_k \rangle$, we deduce from the variational characterization of the eigenvalues $\lambda_1$, $\lambda_k$ and $\lambda_{k+1}$ that

$$\mathcal{J}(u) \le \frac{1}{2}\left(1 - \frac{\mu}{\lambda_k}\right)\|u\|^2 + C_1|\Omega| + \frac{\|h\|_{L^2}}{\lambda_1}\|u\|, \quad \forall u \in V$$

and

$$\mathcal{J}(u) \ge \frac{1}{2}\left(1 - \frac{\overline{\mu}}{\lambda_{k+1}}\right)\|u\|^2 - C_2|\Omega| - \frac{\|h\|_{L^2}}{\lambda_1}\|u\|, \quad \forall u \in V^\perp.$$

So, it is possible to choose $R > 0$ such that

$$\max_{u \in V, \|u\| = R} \mathcal{J}(u) < \inf_{u \in V^\perp} \mathcal{J}(u).$$

All the hypotheses of the Rabinowitz saddle point Theorem 5.3.9 are satisfied, and the variational proof of Theorem 8.3.3 is also concluded in Case 3. □

## 8.4 Positive Solutions

Here we will be interested in applying Theorem 8.3.3 to find positive solutions of

$$\begin{cases} -\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{8.9}$$

under the assumption that $f(x,0) \ge 0$. In such a case we will define $f$ for $u < 0$ by setting

$$f(x,u) \equiv f(x,0), \quad \forall\, u \le 0. \tag{8.10}$$

Since the modified $f$ satisfies $f(x,u) \ge 0$ for every $u \le 0$, then by the maximum principle, any solution of (8.9), with the modified $f$, is greater than or equal to zero so that the value of $f$ for $u \le 0$ does not play any role. In the sequel we will always understand that $f$ denotes the given nonlinearity extended to negative $u$ by (8.10).

If $\gamma_+ := \lim_{s \to +\infty} f(x,s)/s < \lambda_1$, then Theorem 8.3.3 gives us the existence of a (non-negative) solution of (8.9) obtained as a global minimum of the associated Euler functional. In order to prove that this is not zero we need some additional hypotheses on the behavior of $f$ at zero. Notice that the role played by $-\infty$ in the study of the existence of solutions in the previous sections will be now replaced by $u = 0$. In this way, the following result is the counterpart of Theorem 8.3.3 for positive solutions in the case $\max\{\gamma_-, \gamma_+\} < \lambda_1$.

**Theorem 8.4.1** *Assume that $\gamma_+ < \lambda_1$ and let us suppose that*

$$\gamma_0 := \lim_{u \to 0^+} \frac{f(x,u)}{u} > \lambda_1. \tag{8.11}$$

*Then (8.9) has a strictly positive solution.*

*Proof* Let $z$ be the minimum found in Theorem 8.3.3. By the maximum principle, $z \ge 0$. If (8.11) holds, let us show that $z \ne 0$. Actually, let $\varphi_1 > 0$ denote the positive eigenfunction corresponding to $\lambda_1$, such that $\|\varphi_1\| = 1$, and evaluate $\mathcal{J}(t\varphi_1)$ for $t \sim 0$. Fixing $\varepsilon > 0$ such that $\gamma_0 - \varepsilon > \lambda_1$, by (8.11) there exists $\delta > 0$ such that $f(u) \geq (\gamma_0 - \varepsilon)u$ provided $0 < u < \delta$. Then $F(x,u) \geq \frac{1}{2}(\gamma_0 - \varepsilon)|u|^2$ for $0 < u < \delta$ and if $t < \delta \|\varphi_1\|_\infty^{-1}$ we find

$$\mathcal{J}(t\varphi_1) = \tfrac{1}{2}t^2 - \int F(x, t\varphi_1)dx \leq \tfrac{1}{2}t^2 - \tfrac{1}{2}t^2(\gamma_0 - \varepsilon)\int \varphi_1^2 dx.$$

Since $-\Delta\varphi_1 = \lambda_1\varphi_1$, then $\int \varphi_1^2 dx = \lambda_1^{-1}$ and

$$\mathcal{J}(t\varphi_1) \leq \tfrac{1}{2}t^2\left(1 - \frac{\gamma_0 - \varepsilon}{\lambda_1}\right),$$

which together with $\gamma_0 - \varepsilon > \lambda_1$ implies that

$$\lim_{t \to 0+} \frac{\mathcal{J}(t\varphi_1)}{t^2} < 0,$$

and hence $\mathcal{J}(t\varphi_1) < 0$ provided $t > 0$ is sufficiently small. As a consequence $\mathcal{J}(z) = \min_E \mathcal{J} < 0$, proving that $z \neq 0$. □

Next we give a new proof of the preceding theorem by using sub- and super-solutions. As usual, $\varphi_1 > 0$ denotes a positive eigenfunction corresponding to $\lambda_1$. In addition, let $e > 0$ satisfy

$$\begin{cases} -\Delta e = 1, & \text{in } \Omega \\ \quad e = 1, & \text{on } \partial\Omega. \end{cases}$$

*Another proof of Theorem 8.4.1* In order to apply Theorem 7.2.1 we have to prove the existence of an ordered pair of sub- and super-solutions of (8.9). This is done in two steps.

- Step 1. $v = \varepsilon\varphi_1$ is a sub-solution of (8.9), provided $\varepsilon > 0$ is sufficiently small.
- Step 2. $w = Me$ is a super-solution of (8.9), provided $M$ is sufficiently large.

*Step* 1. Using the assumption $\gamma_0 > \lambda_1$, there exists $\delta > 0$ such that $f(t) > \lambda_1 t$ for all $t \in (0, \delta)$. Then, if $\varepsilon > 0$ is such that $\varepsilon|\varphi_1|_\infty < \delta$, we get $f(\varepsilon\varphi_1) > \lambda_1\varepsilon\varphi_1$ and thus

$$-\Delta v = -\varepsilon\Delta\varphi_1 = \lambda_1\,\varepsilon\,\varphi_1 \leq f(x, \varepsilon\varphi_1) = f(x, v), \quad v_{|\partial\Omega} = 0,$$

proving that $v$ is a sub-solution of (8.9).

*Step* 2. Fixing $\delta > 0$ such that $\delta|e|_\infty < 1$, $\gamma_+ < \lambda_1$ implies that $f(x, Me) \leq \delta Me$ provided $M > 0$ is sufficiently large. Then $w = Me$ satisfies

$$-\Delta w = M \geq M\delta|e|_\infty \geq \delta Me \geq f(x, Me), \quad w_{|\partial\Omega} = M,$$

proving that $w$ is a super-solution.

Moreover, by the Hopf lemma (see [58, Lemma 3.4]), there exists $M > 0$ such that $v = \varepsilon\varphi_1 < w = Me$ in $\Omega$. Therefore, Theorem 7.2.1 applies and (8.9) has a solution $u$ such that $0 < \varepsilon\varphi_1 \leq u \leq Me$. □

*Remark 8.4.2* If $f$ assumes negative values, e.g., if $\lim_{u\to+\infty} f(x,u) < 0$, then, according to the discussion at the proof of Theorem 8.4.1 based on sub- super-solutions, any $w \equiv b > 0$ such that $f(b) \le 0$ can be taken as a super-solution instead of $Me$. For $\varepsilon > 0$ sufficiently small, one has that $\varepsilon\varphi_1 < b$ and (8.9) has a solution $u$ such that $0 < \varepsilon\varphi_1 \le u \le b$.

*Remark 8.4.3* Let us point out that, taking into account Lemma 7.2.6, the two proofs of Theorem 8.4.1 are similar.

**Theorem 8.4.4** *If, in addition to the hypotheses of the preceding theorem,* $f(x,u) = f(u)$ *and* $g(u) := u^{-1} f(u)$ *is decreasing for* $u > 0$*, then* (8.9) *has a unique positive solution.*

*Proof* The uniqueness result is based upon the following lemma which is interesting in itself.

**Lemma 8.4.5** *Suppose that* $g(u)$ *is decreasing for* $u > 0$ *and let* $v, w$ *be a positive sub-solution, resp. super-solution, of* (8.9)*, satisfying* $v = w = 0$ *on* $\partial\Omega$*. Then* $v \le w$ *in* $\Omega$.

*Proof* From the definition of sub- and super-solution we deduce

$$\begin{aligned} -v\Delta w + w\Delta v &\ge v f(w) - w f(v) \\ &= vwg(w) - wvg(v) = vw[g(w) - g(v)]. \end{aligned} \tag{8.12}$$

Let $\chi(t)$ be smooth, nondecreasing and such that

$$\begin{cases} \chi(t) \equiv 0, & \text{if } t \le 0, \\ \chi(t) \equiv 1, & \text{if } t \ge 1, \end{cases}$$

and set, for $\varepsilon > 0$,

$$\chi_\varepsilon(t) = \chi\left(\frac{t}{\varepsilon}\right).$$

Then (8.12) yields

$$\int [-v\Delta w + w\Delta v]\chi_\varepsilon(v - w) = \int vw(g(w) - g(v))\chi_\varepsilon(v - w). \tag{8.13}$$

Let us evaluate the integral $I_{v,w,\varepsilon}$ on the left-hand side of (8.13). Integrating by parts (remember that $v = w = 0$ on $\partial\Omega$) we find

$$\begin{aligned} I_{v,w,\varepsilon} &= \int v\chi_\varepsilon'(v - w)\nabla w \cdot (\nabla v - \nabla w) - \int w\chi_\varepsilon'(v - w)\nabla v \cdot (\nabla v - \nabla w) \\ &= \int v\chi_\varepsilon'(v - w)(\nabla w - \nabla v) \cdot (\nabla v - \nabla w) \\ &\quad + \int (v - w)\chi_\varepsilon'(v - w)\nabla v \cdot (\nabla v - \nabla w). \end{aligned}$$

Since $v > 0$ and $\chi_\varepsilon' \ge 0$ we get

$$I_{v,w,\varepsilon} \le \int (v-w)\chi_\varepsilon'(v-w)\nabla v \cdot (\nabla v - \nabla w).$$

Consider the last integral and set

$$\gamma_\varepsilon(t) = \int_0^t s\chi_\varepsilon'(s)ds.$$

With this notation, one has

$$\int (v-w)\chi_\varepsilon'(v-w)\nabla v \cdot (\nabla v - \nabla w) = \int \nabla v \cdot \nabla(\gamma_\varepsilon(v-w)).$$

Moreover, an integration by parts yields

$$\int \nabla v \cdot \nabla(\gamma_\varepsilon(v-w)) = -\int \gamma_\varepsilon(v-w)\Delta v.$$

Since $0 \le \gamma_\varepsilon(t) \le \varepsilon$ for all $t \in \mathbb{R}$, we infer that

$$-\int \gamma_\varepsilon(v-w)\Delta v \le \int f(v)\gamma_\varepsilon(v-w) \le c\,\varepsilon,$$

for some $c > 0$. In conclusion, putting together the previous inequalities we get

$$I_{v,w,\varepsilon} \le c\,\varepsilon.$$

Inserting this bound in (8.13) we find

$$\int vw(g(w) - g(v))\chi_\varepsilon(v-w) \le c\,\varepsilon.$$

Passing to the limit as $\varepsilon \to 0$ and taking into account the definition of $\chi$, we deduce

$$\int_{\{x\in\Omega: v(x)>w(x)\}} vw(g(w) - g(v)) \le 0.$$

On the other hand, since $g(u) = u^{-1}f(u)$ is decreasing, then

$$g(w) > g(v), \quad \forall\, x \in \Omega : v(x) > w(x).$$

Since $v$ and $w$ are positive, it follows that

$$\int_{\{x\in\Omega: v(x)>w(x)\}} vw(g(w) - g(v)) \ge 0.$$

Then

$$\int_{\{x\in\Omega: v(x)>w(x)\}} vw(g(w) - g(v)) = 0,$$

and we conclude that the set $\{x \in \Omega : v(x) > w(x)\}$ has zero Lebesgue measure. This means that $v \le w$ in $\Omega$, completing the proof. □

*Proof of Theorem 8.4.4 completed* Let $u_1, u_2$ be any pair of positive solutions to (8.9). Applying Lemma 8.4.5 with $v = u_1$ and $w = u_2$ we get that $u_1 \leq u_2$. On the other hand, we can also take $v = u_2$ and $w = u_1$ yielding $u_1 \geq u_2$. Therefore, $u_1 = u_2$, proving the theorem. □

*Example 8.4.6* (i) For $0 < q < 1$, consider the problem

$$\begin{cases} -\Delta u = \lambda u^q, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{8.14}$$

In this case, $g(u) = \lambda u^{q-1}$,

$$\gamma_0 = \lim_{u \to 0^+} g(u) = +\infty, \quad \gamma_+ = \lim_{u \to +\infty} g(u) = 0,$$

and thus Theorems 8.4.1 and 8.4.4 yield the existence of a unique positive solution of (8.14) for all $\lambda > 0$.

(ii) As a second application we can consider the problem

$$\begin{cases} -\Delta u = \alpha u - u^p, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{8.15}$$

with $p > 1$. Here $g(u) = \alpha - u^{p-1}$ and

$$\gamma_0 = \lim_{u \to 0^+} g(u) = \alpha, \quad \gamma_+ = \lim_{u \to +\infty} g(u) = -\infty.$$

Hence Theorem 8.4.4 applies provided $\alpha > \lambda_1$ and yields a unique positive solution $u_\alpha$ of (8.15). Using Proposition 8.1.1, (8.15) has a trivial solution only provided that $0 \leq \alpha \leq \lambda_1$. Furthermore, in the present case $b_\alpha = (\alpha)^{1/(p-1)}$ is a super-solution of (8.15) because $f(b_\alpha) = 0$. Then (8.15) has a solution $u_\alpha$ such that $\varepsilon\varphi_1 \leq u_\alpha < b_\alpha$, see Remark 8.4.2. Since $b_\alpha \to 0$ as $\alpha \to 0$, from $u_\alpha(x) < b_\alpha$ it follows that $|u_\alpha|_\infty \to 0$ as $\alpha \to 0$. This means that $\lambda_1$ is a bifurcation point from the trivial solution. □

Following with the last remark, we wish to see that from $\lambda_1$ there branches off a curve of positive solutions of (8.15). Actually, if we consider the problem

$$\begin{cases} -\Delta u = \lambda f(u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{8.16}$$

for a function $f \in C^1(\mathbb{R}^+)$ such that

$$\gamma_+ \leq 0 \quad \text{and} \quad \lim_{u \to 0^+} g(u) = \gamma_0 \tag{8.17}$$

where $g(u) = f(u)/u$, Theorem 8.4.1 yields a positive solution $u_\lambda$ of (8.16) provided

$$\lambda > \nu := \begin{cases} 0, & \text{if } \gamma_0 = +\infty; \\ \lambda_1 \gamma_0^{-1}, & \text{if } 0 < \gamma_0 < +\infty. \end{cases}$$

**Lemma 8.4.7** *Suppose that $f \in C^1(\mathbb{R}^+)$ is such that*

$$f'(u) < g(u), \quad \forall u > 0 \tag{8.18}$$

*and let $u_\lambda$ be a positive solution of* (8.16). *Then $\lambda_1[f'(u_\lambda)] > 1$, where $\lambda_1[m]$ denotes the first positive eigenvalue of*

$$\begin{cases} -\Delta u = \lambda m(x)u, & \text{in } \Omega \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases}$$

*Proof* The function $u_\lambda$ satisfies $-\Delta u_\lambda = \lambda g(u_\lambda)u_\lambda$, which implies that $\lambda = \lambda_k[g(u_\lambda)]$ for some $k \in \mathbb{N}$. Since $u_\lambda > 0$, it follows that $k = 1$, namely $\lambda = \lambda_1[g(u_\lambda)]$. By assumption, $f'(u_\lambda) < g(u_\lambda)$ and hence the properties of the eigenvalues of (8.14) (see Proposition 1.3.11) imply that $\lambda_1[f'(u_\lambda)] > \lambda_1[g(u_\lambda)] = 1$. □

**Theorem 8.4.8** *Suppose that $f \in C^1(\mathbb{R}^+)$ satisfies* (8.17) *and* (8.18). *Then the family $\{u_\lambda : \lambda > \nu\}$ is a curve.*

*Proof* Let $X = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) : u(x) = 0, \forall x \in \partial\Omega\}$, $Y = C(\Omega)$ and consider the map $F(\lambda, u) = \Delta u + \lambda f(u)$. One has that $F(\lambda, u_\lambda) = 0$ for all $\lambda > \lambda_0$ and the linearized equation $d_u F(\lambda, u_\lambda)[\phi] = 0$ is the problem

$$\begin{cases} -\Delta \phi = \lambda f'(u_\lambda)\phi, & \text{in } \Omega \\ \quad \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{8.19}$$

According to Lemma 8.4.7, $\lambda_1[f'(u_\lambda)] > 1$ and therefore (8.19) has only the trivial solution $\phi = 0$. This allows us to apply the implicit function theorem to $F(\lambda, u) = 0$, showing that the family $\{u_\lambda : \lambda > \nu\}$ is a curve. □

*Remark 8.4.9* Theorem 8.4.8 applies to the Examples 8.4.6 (Eq. (8.15) is not in the form (8.16), but in such a case the proof of Theorem 8.4.8 can be carried out with uninfluential changes), proving that the family $\{u_\lambda : \lambda > 0\}$, resp. $\{u_\lambda : \lambda > \lambda_1\}$ is a curve of solutions of (8.14), resp. (8.15). In the latter case, taking also into account that $|u_\lambda|_\infty \to 0$ as $\lambda \to 0$ (see Example 8.4.6-($i$)), we can draw a diagram representing the family $\{u_\lambda : \lambda > 0\}$ (see Fig. 8.1).

Our last result deals with the case $f(x, 0) > 0$. Since we use the abstract results in Sect. 4.4, like in the preceding examples, it is convenient to introduce a real parameter $\lambda$. Hence, instead of (8.9) , we consider the problem

$$\begin{cases} -\Delta u = \lambda f(x, u), & \text{in } \Omega \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{8.20}$$

**Theorem 8.4.10** *Let us suppose that*

$$f \in C(\overline{\Omega} \times \mathbb{R}^+), \quad f(x, 0) > 0 \quad \forall x \in \Omega, \tag{8.21}$$

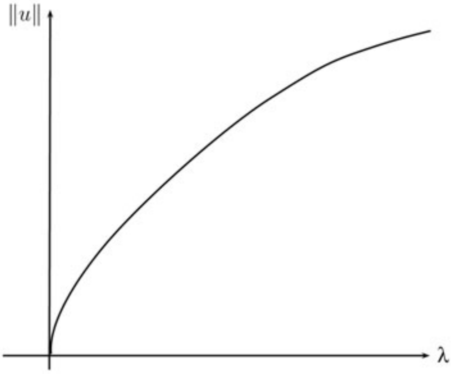



**Fig. 8.1** Bifurcation diagram for Remark 8.4.9

*and* $\lim_{u\to+\infty} f(x,u)/u = \gamma_+ \in \mathbb{R}$. *Then* (8.20) *has a global branch S of positive solutions emanating from* $(0,0)$. *Furthermore, the projection* $Proj_\lambda S$ *of S in the* $\lambda$*-axis contains* $[0,\Lambda)$, *where*

$$\Lambda = \begin{cases} \gamma_+ \cdot \lambda_1, & \text{if } \gamma_+ > 0; \\ \text{is arbitrary}, & \text{if } \gamma_+ \le 0. \end{cases}$$

*In particular, if* $\gamma_+ \le 0$, *for all* $\lambda > 0$ (8.20) *has a positive solution* $u_\lambda$ *such that* $(\lambda, u_\lambda) \in S$.

*Proof* Consider $\Phi(\lambda, u) = u - \lambda K f(u)$. Since $Kf(0) \neq 0$, then Theorem 4.4.1 applies and yields a global branch $S$ of positive solutions emanating from $(0,0)$.

If (8.21) holds, then Proposition 8.2.1 can be used to deduce that $S \cap [0, \Lambda - \varepsilon] \times X$ is bounded for every $\varepsilon > 0$. □

*Remark 8.4.11* Following the ideas of the previous theorem, an additional proof of Theorem 8.4.1 can be given using the Global bifurcation Theorem 6.3.1 instead of Theorem 4.4.1.

# Chapter 9
# Asymptotically Linear Problems

From now on we consider problems which do not possess a priori estimates of their solutions. Specifically, this chapter deals with asymptotically linear problems. For this class of equations it is quite natural to use the bifurcation from infinity. The classical Landesman–Lazer existence result is found by this method as well as by using a variational approach. The bifurcation from infinity also leads to proving the anti-maximum principle.

## 9.1 Existence of Positive Solutions

We give here some of the existence results in the work [14]. Specifically, we study the existence of positive solutions of the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda f(u), \quad & x \in \Omega \\ u &= 0, \quad & x \in \partial\Omega, \end{aligned} \tag{9.1}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $\lambda > 0$ and $f \in C^1([0,+\infty))$, with $f(0) = 0$ and with positive right derivative $f'_+(0) = m_0 > 0$.

First, as in Sect. 8.4, we reduce the study of the existence of positive solutions to the existence of solutions of an extended problem. Indeed, we extend $f$ to $(-\infty, 0)$ by defining $f(s) = f(0)$ for $s < 0$. With this extension, the maximum principle implies that every nontrivial solution of (9.1) is positive.

Now, take $X = C(\overline{\Omega})$, $K : X \longrightarrow X$ the inverse of the Laplacian operator and consider the operator $\Phi : [0,\infty) \times X \longrightarrow X$ given by $\Phi(\lambda, u) = u - \lambda K[f(u)]$, for every $\lambda \geq 0$ and $u \in X$. As in Sect. 7.1, we can rewrite the extended problem (9.1) as the zeros of $\Phi$, i.e.

$$\Phi(\lambda, u) = 0.$$

**Theorem 9.1.1** *If $f(0) = 0$ and $f'_+(0) = m_0 > 0$, then $\lambda_0 = \lambda_1/m_0$ is the unique bifurcation point from zero of positive solutions of* (9.1). *In addition, the continuum emanating from $(\lambda_0, 0)$ is unbounded.*

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82, DOI 10.1007/978-0-8176-8114-2_9, 

*Proof* To apply Theorem 6.3.1 we just have to prove the change of index of $\Phi(\lambda, \cdot)$ as $\lambda$ crosses $\lambda = \lambda_0$. The proof is based on the following claims.

- Claim 1. There exists $\lambda_0 > 0$ such that for every interval $\Lambda \subset [0, +\infty) \setminus \{\lambda_0\}$ there is $\varepsilon > 0$ satisfying

$$\Phi(\lambda, u) \neq 0, \quad \forall \lambda \in \Lambda, \quad \forall 0 < \|u\| < \varepsilon.$$

- Claim 2. For every $\lambda > \lambda_0$ there exists $\delta > 0$ such that

$$\Phi(\lambda, u) \neq \tau\varphi_1, \quad \forall 0 < \|u\| < \delta, \quad \forall \tau \geq 0.$$

To prove Claim 1, we argue by contradiction assuming that there exists a sequence $(\lambda_n, u_n) \in \Lambda \times X$ satisfying

$$\lambda_n \longrightarrow \lambda \neq \lambda_0, \quad \|u_n\| \longrightarrow 0,$$

$$\Phi(\lambda_n, u_n) = 0, \quad u_n \geq 0.$$

Since $K$ is compact, dividing the equation $u_n = \lambda_n K\,[f(u_n)]$ by $\|u_n\|$, we deduce that, up to a subsequence, $u_n \|u_n\|^{-1}$ strongly converges to some $v \in X$. Necessarily, $v$ is an eigenfunction of norm one associated to $\lambda$, i.e., it satisfies

$$v = \lambda K[m_0 v], \quad \|v\| = 1.$$

In particular, $v > 0$. Using $\varphi_1$ as a test function in this eigenvalue problem we obtain

$$\lambda_1 \int v\varphi_1 = \lambda m_0 \int v\varphi_1,$$

and we conclude that $\lambda_1 = \lambda m_0$, which is a contradiction and the proof of Claim 1 is finished.

As a consequence of Claim 1 we obtain

(a) The unique possible bifurcation point of positive solutions is $\lambda = \lambda_0$.
(b) If $\lambda < \lambda_0$ and we take $\Lambda = [0, \lambda]$ then

$$\mathrm{i}\,(\Phi_\lambda, 0) = \mathrm{i}\,(\Phi_0, 0) = \mathrm{i}\,(I, 0) = 1.$$

With respect to the proof of Claim 2, we fix $\lambda > \lambda_0$ and we assume, by contradiction, that there exist sequences $u_n \in X$ and $\tau_n \geq 0$ satisfying $u_n > 0$ in $\Omega$, $\|u_n\| \longrightarrow 0$ and

$$\Phi(\lambda, u_n) = \tau_n\varphi_1,$$

or, equivalently,

$$u_n = \lambda K[f(u_n)] + \tau_n\varphi_1.$$

Dividing this equation by $\|u_n\|$ and using the compactness of $K$, we deduce that, up to a subsequence, $K[f(u_n)/\|u_n\|]$ is convergent and hence $\tau_n/\|u_n\|$ is bounded.

Passing again to a subsequence, if necessary, we can assume that $\tau_n/\|u_n\| \longrightarrow \tau \geq 0$ and $u_n/\|u_n\| \longrightarrow v$ with $v \in X$ satisfying

$$\begin{aligned} -\Delta v &= \lambda f'(0)v + \tau\lambda_1\varphi_1, \quad && x \in \Omega \\ v &= 0, && x \in \partial\Omega \\ \|v\| &= 1. \end{aligned}$$

As in Claim 1, we deduce then that $\lambda f'(0) = \lambda_1$, a contradiction.
As a by-product of Claim 2, if $\lambda > \lambda_0$, we derive that

$$\mathrm{i}\,(\Phi_\lambda, 0) = \mathrm{i}\,(\Phi_\lambda - \tau\varphi_1, 0), \quad \forall \tau > 0.$$

But, again using Claim 2, the problem

$$\begin{aligned} -\Delta w &= \lambda f(w) + \tau\varphi_1, \quad && x \in \Omega \\ w &= 0, && x \in \partial\Omega \end{aligned}$$

has no nontrivial solution. Since, $w = 0$ is not a solution provided that $\tau > 0$, we deduce that the last index is zero, i.e.,

$$\mathrm{i}\,(\Phi_\lambda, 0) = 0, \quad \forall \lambda > \lambda_0,$$

and we have proved the change of index. □

## 9.2 Bifurcation from Infinity

**Definition 9.2.1** $\lambda_\infty$ is a bifurcation point from infinity of $\Phi(\lambda, u) = 0$ if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times X$ satisfying

$$\lambda_n \longrightarrow \lambda_\infty, \quad \|u_n\| \longrightarrow +\infty, \quad \Phi(\lambda_n, u_n) = 0.$$

Assume that

$$\Phi(\lambda, u) = u - T(\lambda, u),$$

with $T$ a compact operator. Following [75], if we make the Kelvin transform

$$z = \frac{u}{\|u\|^2}, \quad u \neq 0,$$

we derive that

$$\left.\begin{array}{c} \Phi(\lambda, u) = 0 \\ u \neq 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} z - \|z\|^2 T\left(\lambda, \dfrac{z}{\|z\|^2}\right) = 0, \\ \qquad\qquad z \neq 0. \end{array}\right.$$

Therefore, if we define

$$\widetilde{\Phi}(\lambda, z) = \begin{cases} z - \|z\|^2 T\left(\lambda, \dfrac{z}{\|z\|^2}\right), & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

we deduce that $\lambda_\infty$ is a bifurcation point from infinity for $\Phi(\lambda, u) = 0$ if and only if $\lambda_\infty$ is a bifurcation point from zero for $\widetilde{\Phi}(\lambda, z) = 0$.

**Theorem 9.2.2** *Let $\Omega \subset \mathbb{R}^N$ be bounded and open and let $f$ be a $C^1$ function in* $[0, +\infty)$ *such that*

$$f(s) = m_\infty s + g(s)$$

*where $g$ satisfies*

$$\lim_{s \to +\infty} g(s)/s = 0.$$

*Then $\lambda_\infty = \lambda_1/m_\infty$ is the unique bifurcation point from infinity of positive solutions of* (9.1)*. Moreover, there exists a subset $\Sigma_\infty$ in $\mathbb{R} \times C(\overline{\Omega})$ of positive solutions of* (9.1) *such that $\widetilde{\Sigma}_\infty = \{(\lambda, z) : (\lambda, z/\|z\|^2) \in \Sigma_\infty\} \cup \{(\lambda_\infty, 0)\}$ is connected and unbounded.*

*Proof* The result follows using the same arguments in the proof of Theorem 9.1.1. □

*Remark 9.2.3* Assume that the hypotheses of Theorems 9.1.1 and 9.2.2 are satisfied.

1. Let $\alpha$ be a positive number. If $f(s) > \alpha s$ for every $s > 0$ then it is easy to show (applying Proposition 8.1.1) that the problem (9.1) has no solution for $\lambda \gg 0$. Then in this case, the continuum bifurcating from $(\lambda_0, 0)$ is the same that emanates from infinity at $\lambda_\infty$. See Fig. 9.1.
2. In the case that there exist $0 < \theta_1 < \theta_2$ such that $f(s) \leq 0$, for every $s \in (\theta_1, \theta_2)$, the reader can use the Maximum principle to verify that problem (9.1) has no solution $(\lambda, u)$ in the strip of $\mathbb{R} \times C(\overline{\Omega})$ given by $\theta_1 \leq \|u\|_\infty \leq \theta_2$. Therefore, in this case $\Sigma_\infty \cap \Sigma_0 = \emptyset$. See Fig. 9.2.

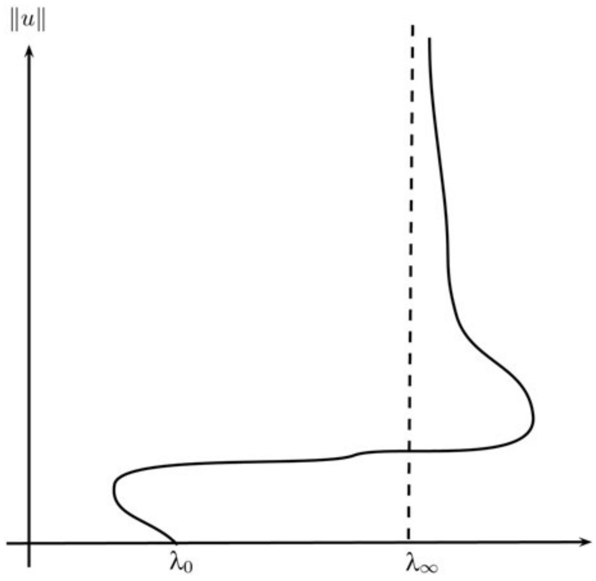



**Fig. 9.1** Bifurcation diagram for Remark 9.2.3-1

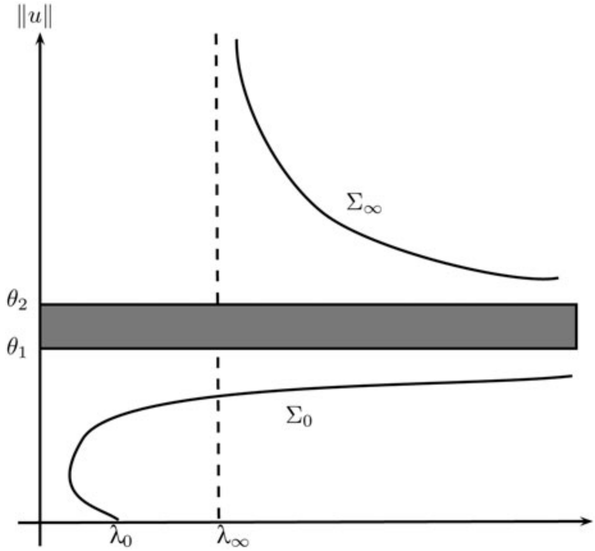



**Fig. 9.2** Bifurcation diagram for Remark 9.2.3-2

## 9.3 On the Behavior of the Bifurcations from Infinity

Let $\Omega \subset \mathbb{R}^N$ be bounded and open and let $g$ be a $C^1$ function in $[0, +\infty)$ satisfying

$$\lim_{s \to +\infty} g(s)/s = 0.$$

Consider the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda u + g(u), \quad & x \in \Omega, \\ u &= 0, & x \in \partial\Omega. \end{aligned} \tag{9.2}$$

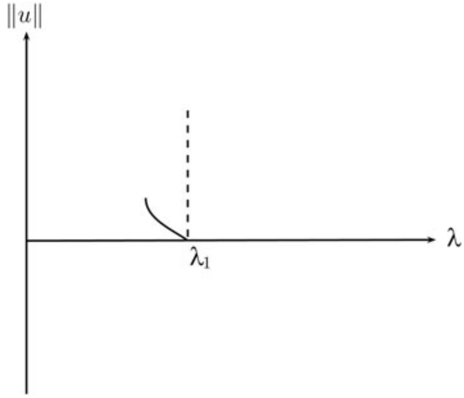
In a similar way to the preceding results, it is possible to prove the following result.

**Theorem 9.3.1** *The value $\lambda_0 = \lambda_1$ is the unique bifurcation point from infinity of positive solutions of* (9.2)*. Moreover, if $g(0) = 0$, then $\lambda = \lambda_1 - g'(0)$ is the unique bifurcation point from zero of positive solutions of* (9.2)*. In addition, there exists a continuum "connecting"* $(\lambda_1 - g'(0), 0)$ *with* $(\lambda_1, \infty)$.

*Proof* The bifurcation from zero at $\lambda_1 - g'(0)$ and the bifurcation from infinity at $\lambda_1$ are deduced as in the preceding theorems. On the other hand, since $g$ is $C^1$, there exists $\alpha > 0$ such that $\alpha u > g(u) > -\alpha u$ for $u > 0$. Then the problem (9.2) has no solution provided that $|\lambda| \gg 0$ and, therefore, the continuum emanating from zero at $\lambda_1 - g'(0)$ is also bifurcating from infinity at $\lambda_1$. □

*Remark 9.3.2* In particular, there exists a solution of (9.2) for every $\lambda$ in the interval of extrema $\lambda_1$ and $\lambda_1 - g'(0)$. However, in the cases that we are able to establish the

**Fig. 9.3** Bifurcation to the left




side of the bifurcations from infinity and from zero, we will improve this existence result.

The side of the bifurcation from zero is completely described by the following theorem. Without loss of generality, we assume that $g'(0) = 0$.

**Theorem 9.3.3** *If $g'(0) = 0$ and there exists $\varepsilon > 0$ such that*

$$g(u) \geq 0, \quad \forall u \in (0, \varepsilon), \tag{9.3}$$

*then the bifurcation from zero of Theorem 9.3.1 is to the left (see Fig. 9.3). Similarly, if the inequality in (9.3) is reversed, then the bifurcation from zero is to the right.*

*Proof* If $(\lambda_n, u_n) \in \mathbb{R} \times X$ are solutions of (9.2) with $\lambda_n \to \lambda_1$ and $\|u_n\| \to 0$, then, as we have seen in the proof of Claim 1 of Theorem 9.1.1, up to a subsequence $u_n/\|u_n\|$ converges to $\varphi_1$. Using this eigenfunction as a test function in the equation satisfied by $u_n$, we obtain

$$(\lambda_1 - \lambda_n) \int_\Omega u_n \varphi_1 = \int_\Omega g(u_n)\varphi_1.$$

Since $0 < u_n$ is uniformly convergent to zero, we deduce by (9.3) that $g(u_n(x)) \geq 0$, for every $x \in \Omega$ and hence that $\lambda_n \leq \lambda_1$.

The result for the reversed inequalities is proved similarly. □

With respect to the bifurcation from infinity we can prove the following result.

**Theorem 9.3.4** *[24] If there exists $\varepsilon > 0$ such that*

$$g(u)u^2 \geq \varepsilon, \quad \forall u \gg 0, \tag{9.4}$$

*then the bifurcation from infinity of the preceding theorem is to the left (see Fig. 9.4). Similarly, if the inequality in (9.4) is reversed, then the bifurcation from infinity is to the right.*

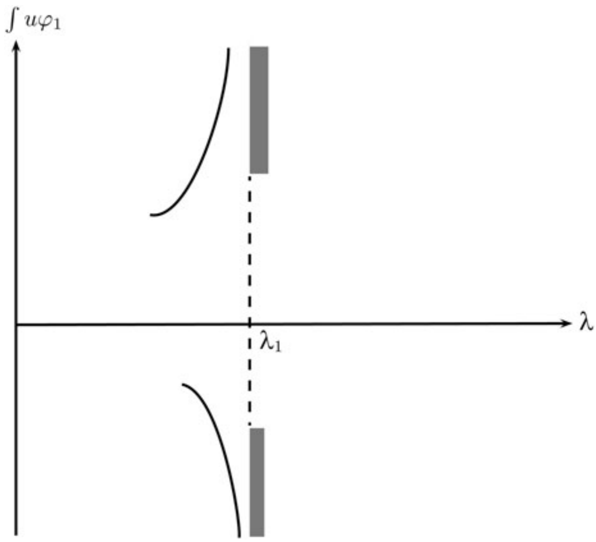



**Fig. 9.4** Bifurcation diagram if (9.4) is satisfied

*Proof* We give here the proof in the case that a more restrictive hypothesis than (9.4) is satisfied, namely we assume that there exists $\alpha < 2$ such that

$$g(u)u^{\alpha} \geq \varepsilon, \quad \forall u \gg 0. \tag{9.5}$$

For the general case we refer to [24].

If $(\lambda_n, u_n) \in \mathbb{R} \times X$ are solutions of (9.2) with $\lambda_n \to \lambda_1$ and $\|u_n\| \to \infty$, then, up to a subsequence, $u_n/\|u_n\|$ converges to $\varphi_1$. Using this eigenfunction as a test function in the equation satisfied by $u_n$ and dividing by $\|u_n\|$, we obtain

$$(\lambda_1 - \lambda_n) \int \frac{u_n}{\|u_n\|}\varphi_1 = \frac{1}{\|u_n\|} \int g(u_n)\varphi_1.$$

Hence, taking into account that $\int \frac{u_n}{\|u_n\|}\varphi_1$ converges to $\int \varphi_1^2 > 0$, we deduce

$$\operatorname{sgn}[\lambda_1 - \lambda_n] = \operatorname{sgn}\left[\int g(u_n)\varphi_1\right].$$

To conclude the proof, we just have to show that the sign of the right-hand side is positive. This is deduced from the Fatou lemma. Indeed, using the fact that $u_n/\|u_n\|$ converges to $\varphi_1$, we deduce that $u_n(x)$ converges to $+\infty$ and, by (9.5), we have

$$\begin{aligned}\liminf_{n\to+\infty} \|u_n\|^{\alpha} \int g(u_n)\varphi_1 &= \liminf_{n\to+\infty} \int g(u_n)u_n^{\alpha}\left(\frac{u_n}{\|u_n\|}\right)^{-\alpha}\varphi_1 \\ &\geq \varepsilon \int {\varphi_1}^{1-\alpha} > 0.\end{aligned}$$

□

*Remark 9.3.5* In [24], some counterexamples show that, in general, if the nonlinearity $g$ is below any quadratic hyperbola $c/s^2$, then the side of the bifurcation from infinity cannot be decided. The case of quasilinear operators in divergence form (instead of the Laplacian operator) is studied in [23]. More recent results can be found in [55, 56].

## 9.4 The Local Anti-Maximum Principle

As a consequence of the preceding results, we point out the bifurcation nature of some classical results like the (local) anti-maximum principle of Clement and Peletier and the Landesman–Lazer theorem for resonant problems.

**Theorem 9.4.1** *Let $r > N$. For every $h \in L^r(\Omega)$, there exists $\varepsilon = \varepsilon(h) > 0$ such that*

1. *If $\int_\Omega h\varphi_1 < 0$, then every solution $(\lambda, u_\lambda)$ of*

$$\begin{aligned} -\Delta u &= \lambda u + h(x), \quad && x \in \Omega, \\ u(x) &= 0, && x \in \partial\Omega, \end{aligned} \tag{9.6}$$

   *satisfies*
   *(a) (**local anti-minimum principle**) $u_\lambda > 0$ in $\Omega$ provided that $\lambda_1 < \lambda < \lambda_1 + \varepsilon$,*
   *(b) (**local minimum principle**) $u_\lambda < 0$ in $\Omega$ provided that $\lambda_1 - \varepsilon < \lambda < \lambda_1$.*
2. *If $\int_\Omega h\varphi_1 > 0$, then every solution $(\lambda, u)$ of (9.6) satisfies*
   *(a) (**local anti-maximum principle**) $u_\lambda < 0$ in $\Omega$ provided that $\lambda_1 < \lambda < \lambda_1 + \varepsilon$,*
   *(b) (**local maximum principle**) $u_\lambda > 0$ in $\Omega$ provided that $\lambda_1 - \varepsilon < \lambda < \lambda_1$.*
3. *If $\int_\Omega h\varphi_1 = 0$, then every solution $(\lambda, u_\lambda)$ of (9.6) with $\lambda \neq \lambda_1$ changes sign in $\Omega$.*

*Remark 9.4.2* In [41, Theorem 2] Clement and Peletier proved a slightly less general version of the cases *1(a)* and *2(a)* of this theorem. Indeed, these authors substituted the condition of the sign of the integral of $u\varphi_1$ by a condition on the sign of $h$ in all $\Omega$.

*Proof* We start with case *1*. Note that by the Fredholm alternative, the linear problem (9.6) has no solution for $\lambda = \lambda_1$, and there is a unique solution if $\lambda$ is not an eigenvalue of the Laplacian operator. In addition, for $X = W^{2,r}(\Omega)$, the value $\lambda = \lambda_1$ is a bifurcation point "from $+\infty$" in the sense that there are solutions $(\lambda, u_\lambda)$ emanating from $\lambda_1$ at infinity such that $u_\lambda/\|u_\lambda\|$ is converging in $W^{2,r}(\Omega) \subset C^1(\overline{\Omega})$ to $\varphi_1$ as $\lambda$ tends to $\lambda_1$. Also, there is a bifurcation "from $-\infty$", i.e., solutions $(\lambda, u_\lambda)$ emanating from $\lambda_1$ at infinity such that $u_\lambda/\|u_\lambda\|$ is converging to $-\varphi_1$ as $\lambda$ tends to $\lambda_1$. (See Fig. 9.5 for the bifurcation diagram.) Now it is immediate to conclude from the preceding section that the bifurcation from $+\infty$ is to the right, while the bifurcation

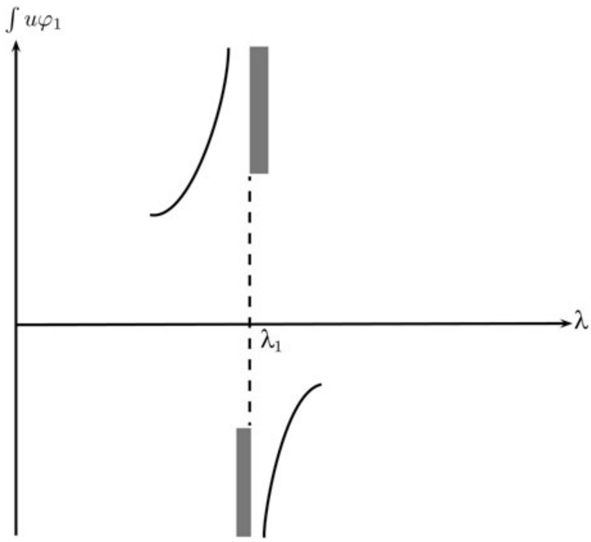



**Fig. 9.5** Bifurcation diagram for case 1 of Theorem 9.4.1

from $-\infty$ is to the left. The proof of *1* is thus concluded. The argument for *2* is similar.

Finally, to prove *3*, it suffices to take $\varphi_1$ as a test function in (9.6) to conclude that every solution $(\lambda, u)$ of this problem satisfies $(\lambda_1 - \lambda)\displaystyle\int_\Omega u\varphi_1 = 0$ and $u$ changes sign. □

*Remark 9.4.3* 1. The choice of $r > N$ allows us to apply our bifurcation results which involve the space $X = W^{2,r}(\Omega)$, continuously embedded in $C^1(\overline{\Omega})$. This fact allows us to ensure that the normalized solutions converge to $\varphi_1$ (or to $-\varphi_1$) in the $C^1$-topology. Since $\varphi_1$ lies in the interior of the cone of positive functions of $C^1(\overline{\Omega})$, then the positivity (or negativity) of the solutions near the bifurcation point easily follows. On the contrary, if we consider $r \le N$, such an argument does not work, and in fact the result is not true, as is proved in [85].
2. A related result for elliptic problems with nonlinear boundary conditions is given in [25].

## 9.5 The Landesman–Lazer Condition

The case of (9.2) with $\lambda = \lambda_1$ is particularly interesting. Thus in this section we study the problem

$$\begin{aligned} -\Delta u &= \lambda_1 u + g(x,u), && x \in \Omega, \\ u(x) &= 0, && x \in \partial\Omega, \end{aligned} \tag{9.7}$$

where $g$ is a bounded Carathéodory function for which

$$\exists g_{+\infty}(x) = \lim_{s\to+\infty} g(x,s) \tag{9.8}$$

$$\exists g_{-\infty}(x) = \lim_{s\to-\infty} g(x,s). \tag{9.9}$$

The classical result by Landesman and Lazer [62] related to resonance at the principal eigenvalue $\lambda_1$ states the following.

**Theorem 9.5.1** *Assume, in addition to* (9.8) *and* (9.9)*, one of the following two conditions:*

$$\int_\Omega g_{+\infty}\varphi_1 < 0 < \int_\Omega g_{-\infty}\varphi_1, \tag{9.10}$$

*or*

$$\int_\Omega g_{+\infty}\varphi_1 > 0 > \int_\Omega g_{-\infty}\varphi_1. \tag{9.11}$$

*Then* (9.7) *admits at least one solution.*

*Proof* We approach the problem (9.7) by embedding it into a one parameter family of problems as follows:

$$\begin{aligned} -\Delta u &= \lambda u + g(x,u), & x \in \Omega,\\ u(x) &= 0, & x \in \partial\Omega, \end{aligned} \tag{9.12}$$

with $\lambda \in \mathbb{R}$. Observe that the boundedness of the function $g$ ensures that bifurcation from infinity for problem (9.12) occurs at $\lambda_1$. In addition, by taking $\varphi_1$ as a test function in (9.12), it is easily deduced that if the condition (9.10) holds then the bifurcation from infinity is to the right. Similarly, if (9.11) holds, the bifurcation from infinity is to the left.

As we will see, the behavior of the bifurcations from infinity at $\lambda_1$ for problem (9.12) determines the existence of solution for the resonant problem (9.7). The key to relate these two problems is to interpret the concepts of bifurcations to the left and to the right in the sense of *a priori bounds* for the norms of the solutions. From this point of view, observe that *every possible bifurcation from $\infty$ at $\lambda_1$ is to the left (resp. to the right) if and only if there exist $\varepsilon > 0$ and $M > 0$ such that every solution $(\lambda, u)$ of* (9.12) *with $\lambda \in [\lambda_1, \lambda_1 + \varepsilon]$ (resp. $\lambda \in [\lambda_1 - \varepsilon, \lambda_1]$) satisfies $\|u\| \le M$.*

Here we just complete the proof in the case that condition (9.11) holds, when the bifurcation from $\infty$ is to the left. In other words, there exists $\varepsilon > 0$ and $M > 0$ such that

$$\|u\| \le M$$

for every solution $(\lambda, u)$ of (9.12) with $\lambda_1 \le \lambda \le \lambda_1 + \varepsilon$.

Taking into account for every $\lambda$ which is not an eigenvalue, that there exists at least a solution $(\lambda, u)$ of (9.12) (see Theorem 8.3.3), we can choose a sequence $(\lambda_n, u_n)$ of solutions with $\lambda_n \to \lambda_1$, $\lambda_n > \lambda_1$, $\forall n \in \mathbb{N}$. Then $\|u_n\| \le M$ for $n$ large and the compactness of $K$ proves that a subsequence of $u_n$ must converge to a solution of the resonant problem (9.7). □

*Remark 9.5.2* The previous general results of the side of bifurcations from infinity can be applied to obtain an improvement of the above classical existence result (see [24]).

## 9.5.1 A Variational Proof of the Landesman–Lazer Result

In this section we complete the study of problem (9.7) by giving another proof of Theorem 9.5.1. This is based on variational techniques and shows that the solutions found are essentially of different variational nature according to the fact that either (9.10) or (9.11) is satisfied.

*Variational proof of Theorem 9.5.1* Consider the functional $\mathcal{J} : H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined for every $u \in H_0^1(\Omega)$ by

$$\mathcal{J}(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{\lambda_1}{2}\int |u|^2 - \int G(x,u),$$

where, as usual $G(x,s) = \int_0^s g(x,t)\,dt$.

**Lemma 9.5.3** *Assume that either* (9.10) *or* (9.11) *is satisfied. Then the functional* $\mathcal{J}$ *satisfies the Palais–Smale condition.*

*Proof* Assume that $\{u_n\}$ satisfies

$$\mathcal{J}(u_n) \le C, \quad \forall n \in \mathbb{N}, \tag{9.13}$$

and

$$|\langle \mathcal{J}'(u_n), v\rangle| \le \epsilon_n \|v\| \quad \forall n \in \mathbb{N}, \quad \forall v \in H_0^1(\Omega) \tag{9.14}$$

with $C > 0$ and $\varepsilon_n$ tending to zero.

The proof will be finished if we prove that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ (see Lemma 7.1.1). Suppose, by contradiction, that $\|u_n\|$ converges to $+\infty$ (up to a subsequence), and define $z_n = u_n/\|u_n\|$. Thus $\{z_n\}$ is bounded in $H_0^1(\Omega)$ and hence, up to subsequences, converges to a function $z$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Dividing (9.13) by $\|u_n\|^2$, we get (using only the fact that $\mathcal{J}(u_n)$ is bounded from above)

$$\limsup_{n\to\infty} \frac{1}{2}\int |\nabla z_n|^2 - \frac{1}{2}\lambda_1 \int |z_n|^2 - \int \frac{G(x,u_n)}{\|u_n\|^2} \le 0.$$

Since, by the hypotheses on $g$ and $\{u_n\}$,

$$\lim_{n\to\infty} \int \frac{G(x,u_n)}{\|u_n\|^2} = 0,$$

while

$$\lim_{n\to\infty} \int |z_n|^2 = \int |z|^2,$$

we have

$$\limsup_{n\to\infty} \int |\nabla z_n|^2 \leq \lambda_1 \int |z|^2.$$

Using the weak lower semicontinuity of the norm, and the Poincaré inequality, we get

$$\lambda_1 \int |z|^2 \leq \int |\nabla z|^2 \leq \liminf_{n\to+\infty} \int |\nabla z_n|^2 \leq \limsup_{n\to+\infty} \int |\nabla z_n|^2 \leq \lambda_1 \int |z|^2.$$

Thus, the inequalities are indeed equalities, so that (by the uniform convexity of $H_0^1(\Omega)$) $\{z_n\}$ converges strongly to $z$ in $H_0^1(\Omega)$ and

$$\int |\nabla z|^2 = \lambda_1 \int |z|^2.$$

This implies, by the definition of $\lambda_1$, that $z = \pm\varphi_1$ (observe that the norm of $z$ in $H_0^1(\Omega)$ is 1 by the strong convergence of $\{z_n\}$ to $z$).

Let us write (9.13), and (9.14) with $v = u_n$. We have

$$-c \leq \int |\nabla u_n|^2 - \lambda_1 \int |u_n|^2 - 2 \int G(x, u_n) \leq c,$$

$$-\epsilon_n \|u_n\| \leq -\int |\nabla u_n|^2 + \lambda_1 \int |u_n|^2 + \int g(x, u_n)\, u_n \leq \epsilon_n \|u_n\|.$$

Summing up, and dividing by $\|u_n\|$,

$$\left| \int [g(x, u_n)\, z_n - 2\, h(x, u_n)\, z_n] \right| \leq \frac{c}{\|u_n\|} + \epsilon_n,$$

where

$$h(x, s) = \begin{cases} \dfrac{G(x, s)}{s} & \text{if } s \neq 0, \\ g(x, 0) & \text{if } s = 0. \end{cases} \tag{9.15}$$

Letting $n$ tend to infinity, we get

$$\lim_{n\to+\infty}\int [g(x,u_n)\,z_n - 2\,h(x,u_n)\,z_n] = 0.$$

Suppose that, for example, $z_n$ converges to $+\varphi_1$. Then $u_n(x)$ tends to $+\infty$ for almost every $x \in \Omega$, and so, by (9.8)

$$g(x,u_n(x)) \to g_{+\infty}(x) \quad \text{for almost every } x \in \Omega,$$

$$h(x,u_n(x)) \to g_{+\infty}(x) \quad \text{for almost every } x \in \Omega.$$

Consequently, the properties of $g$ and $G$, and the Lebesgue theorem imply

$$\lim_{n\to+\infty}\int [g(x,u_n)\,z_n - 2\,h(x,u_n)\,z_n] = -\int g_{+\infty}\,\varphi_1,$$

and so,

$$0 = \int g_{+\infty}\,\varphi_1,$$

which contradicts both (9.10) and (9.11). Thus $\{u_n\}$ is bounded and the lemma follows. □

*Remark 9.5.4* We explicitly remark that in the above proof it is shown that *"for every sequence $u_n$ such that $\mathcal{J}(u_n)$ is bounded from above and $\|u_n\|$ converges to $+\infty$, we have, up to a subsequence, the strong convergence of $u_n/\|u_n\|$ to $\pm\varphi_1$."*

*Variational proof of Theorem 9.5.1 completed* Once we have proved that $\mathcal{J}$ satisfies the Palais–Smale condition, we study the geometry of $\mathcal{J}$ which depends strongly on the conditions (9.10) and (9.11). The proof is divided into two steps.

*Step* 1. If (9.11) holds, we will see that $\mathcal{J}$ is coercive and hence (see Theorem 5.4.1) it has a global minimizer, which concludes the proof in this case. Indeed, suppose by contradiction that $\mathcal{J}$ is not coercive, that is, that there exists a sequence $u_n$ such that $\mathcal{J}(u_n)$ is bounded from above and $\|u_n\|$ converges to $+\infty$. Applying Remark 9.5.4, we can assume that $u_n/\|u_n\|$ is strongly convergent to $\pm\varphi_1$ and thus

$$0 = \lim_{n\to\infty}\frac{\mathcal{J}(u_n)}{\|u_n\|^2} = -\lim_{n\to\infty}\int \frac{G(x,u_n)}{\|u_n\|^2}.$$

However, using (9.8) (resp. (9.9)), by the L'Hôpital rule we have

$$\lim_{n\to\infty}\int \frac{G(x,u_n)}{\|u_n\|^2} = \int g_{+\infty}\varphi_1,$$

(resp.

$$\lim_{n\to\infty}\int \frac{G(x,u_n)}{\|u_n\|^2} = \int g_{-\infty}\varphi_1),$$

provided $u_n/\|u_n\|$ converges to $\varphi_1$ (resp. $-\varphi_1$). In any case we obtain a contradiction with the hypothesis (9.11), which proves that $\mathcal{J}$ is coercive and thus the theorem.

*Step* 2. If (9.10) holds, we follow an argument close to that one used in Case 2 of the variational proof of Theorem 8.3.3 given in Sect. 8.3.3, namely we obtain a solution by applying the Theorem 5.3.8 to $\mathcal{J}$. Indeed, again by the L'Hôpital rule, using (9.8) and (9.10), we deduce that

$$\lim_{t\to+\infty} \frac{\mathcal{J}(t\varphi_1)}{t^2\|\varphi_1\|^2} = \lim_{t\to+\infty} \int \frac{G(x,t\varphi_1)}{t^2\|\varphi_1\|^2} = \int g_{+\infty}\varphi_1 < 0.$$

Similarly,

$$\lim_{t\to-\infty} \frac{\mathcal{J}(t\varphi_1)}{t^2\|\varphi_1\|^2} = \lim_{t\to-\infty} \int \frac{G(x,t\varphi_1)}{t^2\|\varphi_1\|^2} = -\int g_{-\infty}\varphi_1 < 0.$$

These two facts imply that

$$\lim_{|t|\to+\infty} \mathcal{J}(t\varphi_1) = -\infty.$$

In addition, the variational characterization of $\lambda_2$ gives us (see Case 2 in the variational proof of Theorem 8.3.3)

$$\inf_{\langle\varphi_1\rangle^\perp} \mathcal{J} > -\infty.$$

Then we have verified the geometrical hypotheses, and we conclude the existence of a mountain pass critical point of $\mathcal{J}$, concluding the proof. □

# Chapter 10
# Asymmetric Nonlinearities

This chapter deals with nonlinear problems with nonlinearities whose behavior at $+\infty$ and $-\infty$ jumps through an eigenvalue of the linear part. Specifically, we come back to the problem

$$\begin{cases} -\Delta u = f(u) + h(x), & x \in \Omega, \\ \quad u = 0, & x \in \partial\Omega, \end{cases} \tag{10.1}$$

where $\Omega$ is a bounded open set in $\mathbb{R}^N$ and $h \in C(\overline{\Omega})$. On $f \in C^1(\mathbb{R})$ we assume:

$$\left.\begin{array}{l} \text{(i) Setting } f(u) = g(u)u \text{ for } u \neq 0, \ \exists \gamma_\pm = \lim_{u\to\pm\infty} g(u). \\ \\ \text{(ii) } 0 < \gamma_- < \lambda_1 < \gamma_+ < \lambda_2. \end{array}\right\} \tag{10.2}$$

We first discuss the case in which the precise solutions number can be found by using the global inversion theorem with singularities stated in Sect. 3.5. Moreover, we show how some multiplicity results can be obtained by using sub- and super-solutions jointly with degree arguments or with variational arguments. In Sect. 10.4 we employ the topological degree to find continua of solutions.

## 10.1 The Approach by Ambrosetti and Prodi

We begin the study of our problem by following the ideas of [16] which use the global inversion theorem with singularities.

**Theorem 10.1.1** *Let $f \in C^2(\mathbb{R},\mathbb{R})$ and suppose that (10.2) holds and that $f''(u) > 0$, for every $u \in \mathbb{R}$. Then $Y := C^{0,\nu}(\overline{\Omega}) = Y_0 \cup Y_1 \cup Y_2$, where*

1. *$Y_1$ is a $C^1$ submanifold of codimension 1 in $Y$ and (10.1) has a unique solution in $X := C^{2,\nu}(\overline{\Omega})$, for every $h \in Y_1$;*
2. *(10.1) has no solution in $X$, for every $h \in Y_0$;*
3. *(10.1) has exactly two solutions in $X$, for every $h \in Y_2$.*

The proof will be deduced by the following lemma. We keep the notation introduced in Sect. 8.3.1. In particular, we let $F(u) = -\Delta u - f(u)$, $u \in X$.

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82,
DOI 10.1007/978-0-8176-8114-2_10, 

**Lemma 10.1.2** (i) *$F$ is proper.*

(ii) *The singular set $\Sigma'$ of $F$ is not empty, closed, connected and every $u \in \Sigma'$ is an ordinary singular point.*

(iii) *For every $h \in \Sigma'$, $F(u) = h$ has a unique solution.*

*Proof* (i) To prove the properness of $F$ we argue in a similar way to the proof of Lemma 8.3.1. We limit ourselves to indicating the changes. As before, if $u_k \in X$ satisfies $F(u_k) = h_k \in Y$ and $h_k$ is bounded, then we claim that $u_k$ is bounded in $C^{0,\nu}(\overline{\Omega})$. Otherwise, up to a subsequence, $z_k := u_k \|u_k\|^{-1}$ converges to some $z \in C^1(\overline{\Omega})$, with $\|z\| = 1$, which solves (8.4), where $a$ is given by (8.5). Since $z \neq 0$, it follows that $\lambda_j(a) = 1$ for some integer $j \geq 1$. Since $a < \lambda_2$ the comparison of the eigenvalues (Proposition 1.3.11-(i)) implies that $\lambda_1(a) = 1$. Then the eigenfunction $z$ is either positive or negative in $\Omega$ and hence $a$ equals either $\gamma_+$ or $\gamma_-$. Since both $\gamma_\pm$ are not eigenvalues of the Laplace operator, we get a contradiction. The rest is as in Lemma 8.3.1.

(ii) Fix $z \in X$ with $z > 0$ and write $u = tz + w$, with $t \in \mathbb{R}$ and $w \in (\mathbb{R}z)^\perp$. Remember that $u \in \Sigma'$ whenever

$$-\Delta v = f'(tz + w)v, \qquad v \in X$$

has a nontrivial solution, namely if $\lambda_1[f'(tz+w)] = 1$. Since $f'' \geq 0$, $f'(tz+w) > f'(sz+w)$ provided $t > s$ and thus, by (Proposition 1.3.11-(i)), $t \mapsto \lambda_1[f'(tz+w)]$ is decreasing. Moreover, by case ii) of the same proposition and from $f'(tz+w) \to \gamma_-$ as $t \to -\infty$, resp. $f'(tz+w) \to \gamma_+$ as $t \to +\infty$, it follows that $\lambda_1[f'(tz+w)] \to \lambda_1/\gamma_-$ as $t \to -\infty$, resp. $\lambda_1[f'(tz+w)] \to \lambda_1/\gamma_+$ as $t \to +\infty$. Then there exists a unique $t^*$ such that $\lambda_1[f'(tz+w)] = 1$. This shows that $\Sigma'$ is not empty and has a Cartesian representation on $(\mathbb{R}z)^\perp$, proving the first part of (ii). Since $u \in \Sigma'$ whenever $\lambda_1[f'(u)] = 1$, there exists $\varphi \in X$, which does not change sign in $\Omega$, such that $\mathrm{Ker}\, dF(u) = \mathbb{R}\varphi$ and $\mathrm{Range}\, dF(u) = \mathrm{Ker}\, \psi$ where $\langle \psi, h\rangle = \int h\varphi$. Since $d^2F(u)[\varphi,\varphi] = f''(u)\varphi^2$ we get

$$\langle \psi, d^2F(u)[\varphi,\varphi]\rangle = \int f''(u)\varphi^3 \neq 0$$

because $f''(u) > 0$, proving that $u$ is an ordinary singular point.

(iii) By contradiction, let $u \neq v$ be singular points such that $F(u) = F(v)$. Setting

$$a(x) = \begin{cases} \dfrac{f(u) - f(v)}{u - v}, & \text{if } u(x) \neq v(x), \\[2ex] f'(u(x)), & \text{if } u(x) = v(x), \end{cases}$$

we find that $z = u - v$ satisfies $-\Delta z = a(x)z$. As before we infer that $\lambda_1[a] = 1$ and $z$ is, say, positive, namely $u > v$. Since $f'' > 0$ and $a < f'(v)$ we deduce that $\lambda_1[f'(v)] < 1 = \lambda_1[a]$, a contradiction with the fact that $v \in \Sigma'$. □

*Proof of Theorem 10.1.1* It suffices to use the previous lemma and apply Theorem 3.5.1. □

*Remark 10.1.3* In the preceding proof or, more specifically, in the verification of case (i) of Lemma 10.1.2, it is essential that zero is the unique solution of the problem (10.3),

$$\left.\begin{aligned} -\Delta v &= \alpha v^+ + \beta v^-, && x \in \Omega \\ v &= 0, && x \in \partial\Omega, \end{aligned}\right\} \tag{10.3}$$

provided that $\alpha$ and $\beta$ lie between two consecutive eigenvalues. The set $\Sigma$ of the pairs $(\alpha, \beta)$ such that (10.3) has nontrivial solutions is called the Fučik spectrum. It was Fučik [53] who gave a complete description of it in the case $N = 1$. With respect to the case $N \geq 2$, in Dancer [43], it is shown that the two lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are isolated in $\Sigma$, and in Gallouët and Kavian [54], it is proved that from each pair $(\lambda_k, \lambda_k)$ emanates a curve $S_{k-1}$ in $\Sigma$. A variational characterization of the curve $S_1$ emanating from $(\lambda_2, \lambda_2)$ is given in De Figueiredo and Gossez [49], where, in addition, it is proved that $S_1$ is asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$. In [28] the description of the spectrum in the radial case is given.

## 10.2 The Approach by Amann–Hess

In the following sections, we study different approaches to (10.1). We anticipate that we will not assume $f'' \geq 0$ but we will only obtain estimates from below of the number of solutions. Specifically, this section is devoted to studying the approach due to Amann and Hess [4]. It combines the method of the sub-super-solutions with degree arguments to prove the existence of solutions for the problem

$$\begin{aligned} -\Delta u &= f(u) + t\varphi(x) + h(x), && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned} \tag{10.4$_t$}$$

where $\varphi, h \in C(\overline{\Omega})$, $\varphi(x) > 0$ for $x \in \Omega$, $f \in C^1(\mathbb{R})$ and there exist the limits $\gamma_\pm = \lim_{u \to \pm\infty} g(u)$ with $g(u) = f(u)/u$, for $u \neq 0$.

First, we need to prove the following lemmas.

**Lemma 10.2.1** *If* (10.2) *is satisfied, then the solutions of* (10.4$_t$) *are uniformly bounded on compact sets of t, i.e., for every compact interval* $\Gamma \subset \mathbb{R}$, *there exists* $R > 0$ *such that every solution u of* (10.4$_t$) *with* $t \in \Gamma$ *satisfies*

$$\|u\|_{C^1} \leq R.$$

*Proof* Suppose on the contrary that $u_n$ is a solution of (10.4$_{t_n}$) with $t_n$ bounded and $\|u_n\|_{C^1} \to \infty$. Using that $t_n/\|u_n\|_{C^1}$ converges to zero we deduce that $z_n = u_n/\|u_n\|_{C^1}$ strongly converges to a nonzero solution $z$ of (8.4). As has been seen in Lemma 10.1.2, this problem has only the zero solution and we obtain a contradiction, proving the lemma. □

Following McKenna–Walter [70] we also prove the following nonexistence result.

**Lemma 10.2.2** *If hypothesis* (10.2) *holds, then there exists* $\varepsilon_0 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_0$ *there is* $t_\varepsilon \in \mathbb{R}$ *such that for every* $t < t_\varepsilon$ *and* $\lambda \in [0, 1]$, *the problem*

$$\begin{aligned} -\Delta u &= \lambda f(u) + t\varphi + h, \quad && x \in \Omega, \\ u &= 0 && x \in \partial\Omega, \end{aligned}$$

*has no solution in* $\partial B_{|t|\varepsilon}(t\phi) = \{u \in C_0^1(\overline{\Omega}) : \|u - t\phi\|_{C^1} = |t|\varepsilon\}$, *where* $\phi$ *denotes the unique solution in* $C_0^1(\overline{\Omega})$ *of* $-\Delta\phi = \varphi$ *in* $\Omega$.

*Proof* Let $\varepsilon_0 > 0$ be such that $\|\phi\|_{C^1} > \varepsilon_0$, $\|\phi\|_2^2 - \varepsilon_0\|\varphi\|_1 > 0$ and $\lambda_2\varepsilon_0\|\varphi\|_1 < \lambda_1^2\left[\|\phi\|_2^2 - \varepsilon_0\|\varphi\|_1\right]$. We argue by contradiction and suppose that for some $\varepsilon \in (0, \varepsilon_0)$ there exist sequences $t_n \in \mathbb{R}$, $\lambda_n \in [0, 1]$ and $u_n \in C_0^1(\overline{\Omega})$ with $t_n \to -\infty$, $\lambda_n \to \lambda \in [0, 1]$ and $\|\frac{u_n}{t_n} - \phi\|_{C^1} = \varepsilon$, satisfying

$$-\Delta u_n = \lambda_n f(u_n) + t_n\varphi + h, \quad x \in \Omega.$$

Since $\|\frac{u_n}{t_n} - \phi\|_{C^1} = \varepsilon$, the sequence $z_n := u_n/t_n$ is bounded. Moreover, $\|u_n\|_{C^1} \to \infty$, because otherwise $z_n \to 0$ in $C_0^1(\overline{\Omega})$ and thus $0 \in B_\varepsilon(\phi)$, which is impossible by the choice of $\varepsilon_0$.

On the other hand, there exists $z \in H_0^1(\Omega)$ such that (up to a subsequence) $z_n \to z$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and $z_n(x) \to z(x)$ a.e. $x \in \Omega$. Arguing as before, we deduce the strong convergence of $z_n$ to $z$. Consequently, $\|z - \phi\|_{C^1} = \varepsilon$.

Dividing by $t_n$ the equation satisfied by $u_n$ and taking limits as $n$ tends to infinity, we deduce from (10.2) that $z$ satisfies the following equation:

$$-\Delta z = \lambda a(x)z + \varphi, \quad x \in \Omega$$

where $a(x)$ is given by (8.5). Since $z \neq \phi$, we have $\lambda \neq 0$. We claim that $z$ is non-negative. Indeed, by taking $z^- := \min\{z, 0\}$ as a test function in the equation satisfied by $z$ and using that $\gamma_- < \lambda_1$, we obtain from Corollary 1.3.9

$$\begin{aligned} \lambda_1\|z^-\|_2^2 \le \int \nabla z \cdot \nabla z^- &= \lambda \int a(x)(z^-)^2 + \int \varphi z^- \\ &= \lambda \int \gamma_-(z^-)^2 + \int \varphi z^- < \lambda\lambda_1\|z^-\|_2^2. \end{aligned}$$

Since $\lambda \in [0, 1]$, then $z^- \equiv 0$, proving the claim.

We now take $\phi$ as a test function in the equation satisfied by $z$ and $z$ in the equation satisfied by $\phi$ to get

$$\int z\varphi = \int \nabla z \cdot \nabla\phi = \lambda\gamma_+ \int z\phi + \int \varphi\phi.$$

We have

$$\|\varphi\|_1\|z - \phi\|_{C^1} \ge \int \varphi(z - \phi) = \lambda\gamma_+ \int z\phi \ge \lambda\lambda_1 \int z\phi,$$

because $\gamma_+ > \lambda_1$. Taking into account that $\|z-\phi\|_{C^1} = \varepsilon$, we can write $z = \phi + \varepsilon z_1$ with $\|z_1\|_{C^1} = 1$. Thus,

$$\varepsilon\|\varphi\|_1 \geq \lambda\lambda_1 \int z\phi = \lambda\lambda_1 \left[\|\phi\|_2^2 + \varepsilon \int z_1\phi\right] \geq \lambda\lambda_1 \left[\|\phi\|_2^2 - \varepsilon\|\varphi\|_1\right].$$

Since $\|\phi\|_2^2 - \varepsilon\|\phi\|_1 > 0$, if $\varepsilon < \varepsilon_0$, this implies that $\lambda \leq \dfrac{\varepsilon\|\varphi\|_1}{\lambda_1\left[\|\phi\|_2^2 - \varepsilon\|\varphi\|_1\right]} < \dfrac{\lambda_1}{\lambda_2}$.

On the other hand, $z$ is a positive super-solution of the problem

$$\begin{aligned} -\Delta u &= \lambda\gamma_+ u \qquad && x \in \Omega, \\ u &= 0 && x \in \partial\Omega. \end{aligned}$$

If $\delta < 1/\lambda_1$, $w = \delta\lambda\varphi_1$ is a sub-solution of this problem. We can choose $\delta$ small enough to conclude that $w \leq z$. The method of sub- and super-solution (Theorem 7.2.1) allows us to deduce the existence of a non-negative, nontrivial solution. As a consequence, because we have previously shown that $\lambda < \frac{\lambda_1}{\lambda_2}$, $\lambda = \frac{\lambda_1}{\gamma_+} > \frac{\lambda_1}{\lambda_2}$, which is a contradiction. □

**Theorem 10.2.3** *If condition* (10.2) *holds, then there exists a number* $t^* \in \mathbb{R}$ *such that the problem* ($10.4_t$) *has*

- *no solution if* $t > t^*$,
- *at least one solution if* $t = t^*$
- *and at least two solutions if* $t < t^*$.

*Proof* We begin by observing that hypothesis (10.2) means that there exist $\delta < \lambda_1 < \overline{\delta}$ and $C > 0$ such that

$$f(s) \geq \delta s - C \tag{10.5}$$

and

$$f(s) \geq \overline{\delta} s - C, \tag{10.6}$$

for $s \in \mathbb{R}$.

We show first that problem ($10.4_t$) has no solution provided that $t$ is large enough. Indeed, this is deduced by taking a first eigenfunction $\varphi_1 > 0$ as a test function to obtain

$$\lambda_1 \int u\varphi_1 = \int f(u)\varphi_1 + t\int \varphi\varphi_1 + \int h\varphi_1.$$

Hence, if $u$ is a solution of ($10.4_t$) with $\int u\varphi_1 \geq 0$, then, by (10.6),

$$t\int \varphi\varphi_1 \leq (\lambda_1 - \overline{\delta})\int u\varphi_1 - C\int \varphi_1 - \int h\varphi_1 \leq -C\int \varphi_1 - \int h\varphi_1$$

and the positiveness of $\varphi$ (and of $\varphi_1$) implies that $t$ is bounded from above. A similar argument using (10.5) (instead of (10.6)) shows the same if $\int u\varphi_1 < 0$.

Consequently, the set $S := \{t \in \mathbb{R} : (10.4_t) \text{ admits a solution}\}$ is bounded from above. The rest of the proof is divided into two steps:

*Step* 1 $S$ is a nonempty closed interval, i.e., there exists $t^*$ such that $S = (-\infty, t^*]$.
*Step* 2 Problem $(10.4_t)$ has at least two solutions for $t < t^*$.

*Proof of Step 1* First we apply Lemma 10.2.2 to prove that $S$ is not the empty set. In order to do that we use the Leray–Schauder degree. Let $t < t_\varepsilon$ (where $t_\varepsilon$ is given by Lemma 10.2.2) and $\Phi_t(u) = u - K(f(u) + t\varphi + h)$ where, as usual, $K : C_0^1(\overline{\Omega}) \longrightarrow C_0^1(\overline{\Omega})$ is the inverse operator of the Laplacian operator in the space $C_0^1(\overline{\Omega})$. By the invariance of the Leray–Schauder degree, Lemma 10.2.2 implies that

$$\deg(\Phi_t, B_{|t|\varepsilon}(t\phi), 0) = \deg(I - K(t\varphi), B_{|t|\varepsilon}(t\phi), 0) = 1.$$

Then there exists a solution of $(10.4_t)$ in $B_{|t|\varepsilon}(t\varphi)$ and the set $S$ is not empty.

Now, we observe that $S$ is an interval unbounded from below. Indeed, if $t_0 \in S$ then there exists a solution $u_0$ of $(10.4_{t_0})$. Clearly, it is a super-solution for $(10.4_t)$ for every $t < t_0$. Moreover, if $\underline{u}_t$ is the unique solution of the linear problem

$$\begin{cases} -\Delta u = \delta u - C + t\varphi(x) + h(x), & x \in \Omega, \\ \quad u = 0, & x \in \partial\Omega, \end{cases}$$

then condition (10.5) implies that $\underline{u}_t$ is a sub-solution of $(10.4_t)$ with $-\Delta\underline{u}_t \leq -\Delta u_0$ (already we have just proved that $\underline{u}_t$ is less than or equal to every super-solution of $(10.4_t)$) and thus, by the maximum principle, $\underline{u}_t \leq u_0$. Therefore, the sub-super-solution method[1] (Theorem 7.2.1) applies, and we conclude that $(-\infty, t_0] \subset S$ and $S$ is an interval unbounded from below.

Let $t^* = \sup S$. To conclude the proof of Step 1, it suffices to show that $t^* \in S$. To this end let $\{t_n\}$ be a sequence in $S$ converging to $t^*$. For every $t_n$, let $u_n$ be a solution of $(10.4_{t_n})$, i.e., $u_n = K(f(u_n) + t_n\varphi + h)$. By Lemma 10.2.1, $\|u_n\|_{C^1}$ is bounded and from the compactness of $K$ we deduce that—up to a subsequence— $u_n$ strongly converges to a solution of $(10.4_t^*)$ and $t^* \in S$.

*Proof of Step 2* Fix $t < t^*$ and let $u^*$ be a solution of $(10.4_{t^*})$. Then, as we have seen in the first step, $u^*$ (resp. $\underline{u}_t$) is a super-solution (resp. a sub- solution) of $(10.4_t)$ with $\underline{u}_t \leq u^*$. Further, by the strong maximum principle and the Hopf lemma (see [58, Lemma 3.4]) we have $\underline{u}_t < u^*$ in $\Omega$ and $\frac{\partial u^*}{\partial \nu} < \frac{\partial u}{\partial \nu}$ on $\partial\Omega$. Thus, we can define the set

$$U_t(R) = \{u \in C_0^1(\overline{\Omega}) : \underline{u}_t < u < u^* \text{ in } \Omega, \ \frac{\partial u^*}{\partial n} < \frac{\partial u}{\partial n} < \frac{\partial \underline{u}_t}{\partial n} \text{ on } \partial\Omega\} \cap B_R(0).$$

Let $\Phi_t(u) = I - K(f(u) + t\varphi + h)$. By Lemma 7.2.3, there is $R > 0$ such that $\deg(\Phi_t, U_t(R), 0) = 1$ which, by the existence property of the degree, implies the existence of a first solution of $(10.4_t)$ in $U_t(R)$. The key idea to find the second solution of $(10.4_t)$ is to compute the degree of $\Phi_t$ in $B_R(0)$ and to use the excision

[1] Indeed, by Remark 2.2.3, we deduce the existence of a minimal solution of $(10.4_t)$.

property. Indeed, by Lemma 10.2.1, if $t_1 > t^*$, $R$ may be chosen such that $\|u\|_{C^1} < R$ for each solution $u$ of $(10.4_s)$ with $s \in [t, t_1]$. Using the homotopy invariance of the Leray–Schauder degree and the fact that problem $(P_{t_1})$ has no solution, we get

$$\deg(\Phi_t, B_R(0), 0) = \deg(\Phi_{t_1}, B_R(0), 0) = 0.$$

Therefore, the excision property of the degree implies that

$$\deg(\Phi_t, B_R(0) \setminus U_t(R), 0) = \deg(\Phi_t, B_R(0), 0) - \deg(\Phi_t, U_t(R), 0) = -1$$

which means that, in addition to the solution of $(10.4_t)$ in $U_t(R)$, there exists a second solution in $B_R(0) \setminus U_t(R)$. Therefore, Step 2 has been proved and thus the theorem. □

## 10.3 Variational Approach by Mountain Pass and Sub- and Super-Solutions

We devote this section to discuss a different proof [51] of the existence of the second solution in Theorem 10.2.3. This is based on variational arguments. Specifically, in the proof given in the previous section we have seen that the set $S = \{t \in \mathbb{R} : (10.4_t) \text{ admits a solution}\} = (-\infty, t^*]$ and that for every $t \in (-\infty, t^*]$ there exist a sub-solution and a super-solution of $(10.4_t)$ which are well ordered. By applying Lemma 6 there exists a solution $u_1$ of $(10.4_t)$ which is (between the sub-solution and the super-solution and) a local minimizer of the functional

$$\mathcal{J}(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(u) - t\int \varphi u - \int hu, \ u \in H_0^1(\Omega),$$

where $F(u) = \int_0^u f$.

In addition, we have

$$\begin{aligned}\lim_{s\to+\infty} \frac{\mathcal{J}(s\varphi_1)}{s^2} &= \lim_{s\to+\infty}\left[\frac{1}{2}\int |\nabla\varphi_1|^2 - \int \frac{F(s\varphi_1)}{s^2} - \frac{t}{s}\int \varphi\varphi_1 - \frac{1}{s^2}\int h\varphi_1\right] \\ &= \frac{1}{2}\int |\nabla\varphi_1|^2 - \frac{\gamma_+}{2}\int \varphi_1^2 \le \frac{1}{2}\left(1 - \frac{\gamma_+}{\lambda_1}\right)\int |\nabla\varphi_1|^2.\end{aligned}$$

Since $\gamma_+ > \lambda_1$ the above estimate implies that

$$\lim_{s\to+\infty} \mathcal{J}(s\varphi_1) = -\infty \tag{10.7}$$

and it is possible to choose an arbitrarily large $s$ such that $\mathcal{J}(s\varphi_1) < \mathcal{J}(u_1)$. In conclusion, the geometry of the mountain pass (Theorem 5.3.6) is satisfied. It remains to show that the Palais–Smale condition holds. For this, it suffices to prove that every sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $\{\mathcal{J}(u_n)\}$ is bounded and $\{\mathcal{J}'(u_n)\}$

tends to zero in $H_0^1(\Omega)$ is bounded in $H_0^1(\Omega)$ (see Lemma 7.1.1). Assume, by contradiction, that $\|u_n\| \to +\infty$ (up to a subsequence) and observe that using that $\lim_{n\to+\infty} \mathcal{J}'(u_n)(\phi)/\|u_n\| = 0$ and taking $v_n \equiv u_n/\|u_n\|$, we obtain

$$\lim_{n\to+\infty}\left[\int \nabla v_n \cdot \nabla\varphi - \int \frac{f(u_n)}{\|u_n\|}\phi - \int \frac{t\varphi\phi}{\|u_n\|} - \int h\frac{\phi}{\|u_n\|}\right] = 0,$$

for every $\phi \in H_0^1(\Omega)$. Passing to a subsequence if necessary, we may assume without loss of generality that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \to v$ in $L^2(\Omega)$, $v_n(x) \to v(x)$ a.e. $x \in \Omega$. Thus, by the Lebesgue dominated convergence theorem and (10.2) we obtain

$$\lim_{n\to+\infty}\int \frac{f(u_n)}{\|u_n\|}\phi = \lim_{n\to+\infty}\int g(u_n)v_n\phi = \int \left(\gamma_+ v^+ + \gamma_- v^-\right)\phi.$$

Hence

$$\int \nabla v \cdot \nabla\phi = \int \left(\gamma_+ v^+ + \gamma_- v^-\right)\phi,$$

i.e., $v$ is a solution of the problem (8.4). As has been seen in Lemma 10.1.2, this implies that $v = 0$, a contradiction because

$$0 = \lim_{n\to+\infty}\mathcal{J}'(u_n)(v_n) = 1 - \lim_{n\to+\infty}\left[\int f(u_n)v_n - t\int \varphi v_n - \int h v_n\right] = 1.$$

Hence, $u_n$ is bounded and the Palais–Smale condition has been verified. Applying Theorem 5.3.6, we obtain the existence of a critical point (and thus a solution of $(10.4_t)$) $u_2 \neq u_1$ of $\mathcal{J}$. The variational proof of the existence of a second solution in Theorem 10.2.3 is thus concluded. □

*Remark 10.3.1* Since we devoted this section to apply variational methods, it is really worthwhile to see that, if $\varphi = \varphi_1$, then the mountain pass theorem may also be applied to prove that the set $S$ of all $t$ for which problem $(10.4_t)$ can be solved is not empty (i.e., a variational proof of Step 1 of the proof of Theorem 10.2.3). Indeed, we are going to show that $(10.4_t)$ is solvable if $t \ll 0$. To this end, consider the subspace $W = \{u \in H_0^1(\Omega) : \int u\varphi_1 = 0\}$ (which is orthogonal to $\mathbb{R}\varphi_1$). Roughly speaking, since the first eigenvalue in $W$ of the Laplacian operator is $\lambda_2$ (remember the variational characterization of $\lambda_2$ given in Theorem 1.3.8) and condition (10.2) holds, we deduce that

$$\inf_{w\in W}\mathcal{J}(w) > -\infty.$$

Since $\varphi = \varphi_1$, we observe that $\mathcal{J}(w) = \frac{1}{2}\int|\nabla w|^2 - \int F(w) - \int hw$ does not depend on $t$ and thus, there is $t_0 \ll 0$ such that $\mathcal{J}(-\varphi_1) = \frac{1}{2}\int|\nabla\varphi_1|^2 - \int F(-\varphi_1) + t\int\varphi_1^2 + \int h\varphi_1 < \inf_{w\in W}\mathcal{J}(w)$ for every $t \le t_0$.

In addition, by (10.7), for every $t \le t_0$ there is $s \gg 0$ (depending on $t$) such that $\mathcal{J}(s\varphi_1) < \inf_{w\in W}\mathcal{J}(w)$. Theorem 5.3.8 applies and proves that $\mathcal{J}$ has a critical point which is a solution of $(10.4_t)$ for $t \le t_0$.

## 10.4 Approach by Degree Giving a Continuum of Solutions

In this section, we follow the ideas of [20, 21] and apply Theorem 4.4.2 to give an alternative proof of Theorem 10.2.3. Specifically, we prove the following result.

**Theorem 10.4.1** *Let $\varphi \in L^\infty(\Omega)$ be a positive function and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a $C^1$ function satisfying (10.2). Let $t^*$ be the supremum of all $t \in \mathbb{R}$ such that problem* $(10.4_t)$ *admits a solution. Then $t^*$ is finite and there exists a continuum $\mathcal{C}$ in $\Sigma \equiv \{(t,u) \in \mathbb{R} \times C_0^1(\overline{\Omega}) :$ u solution of $(10.4_t)\}$ satisfying that*

1. $(-\infty, t^*] \subset Proj_{\mathbb{R}}\, \mathcal{C}$.
2. *For every $t \in (-\infty, t^*)$, the $t$-slice $\mathcal{C}_t = \{u \in C_0^1(\overline{\Omega}) : (t,u) \in \mathcal{C}\}$ contains two distinct solutions of* $(10.4_t)$.

*Remark 10.4.2* As a consequence, we recover the assertion of Theorem 10.2.3: $(10.4_t)$ has, at least, two (resp. one, zero) solutions for $t < t^*$ (resp. $t \le t^*$, $t > t^*$).

*Proof* As we have seen in the proof of Theorem 10.2.3 in Sect. 10.2, $S = \{t \in \mathbb{R} : (10.4_t) \text{ admits a solution}\} = (-\infty, t^*]$. Observe that the family $\Sigma_{t^*}$ of the solutions of $(10.4_{t^*})$ is clearly a compact set in $C_0^1(\overline{\Omega})$. Let $u^*$ be the minimal solution of $(10.4_{t^*})$ and choose $t_0 < t^*$. We have seen in Step 1 of the proof of Theorem 10.2.3 that it is possible to pick a sub-solution $u_{t_0} < u^*$ of $(10.4_{t_0})$ which is not a solution. Clearly $u_{t_0}$ is also a sub-solution and not a solution for $(10.4_t)$ if $t \in [t_0, t^*]$. As in the proof of Theorem 10.2.3, there is $R > 0$ such that, if $\Phi_t(u) = I - K(f(u) + t\varphi + h)$ and

$$U_{t_0}(R) = \{u \in C_0^1(\overline{\Omega}) : \underline{u}_{t_0} < u < u^* \text{ in } \Omega,\ \frac{\partial u^*}{\partial n} < \frac{\partial u}{\partial n} < \frac{\partial \underline{u}_{t_0}}{\partial n} \text{ on } \partial\Omega\} \cap B_R(0),$$

then $\deg(\Phi_t, U_{t_0}(R), 0) = 1$, for every $t \in [t_0, t^*)$.

Applying Theorem 4.4.2 with $X = C_0^1(\overline{\Omega})$, $[a,b] = [t_0, t^*]$, $U = B_R(0)$ and $U_1 = U_{t_0}(R)$, we deduce the existence of a continuum $\mathcal{S}_{t_0}$ in $\Sigma$ such that

$$\mathcal{S}_{t_0} \cap (\{t_0\} \times U_{t_0}(R)) \ne \emptyset,$$

and

$$\mathcal{S}_{t_0} \cap (\{t_0\} \times [B_R(0) \setminus \overline{U_{t_0}(R)}]) \ne \emptyset.$$

In particular, the continuum $\mathcal{S}_{t_0}$ crosses $\{t\} \times \partial U_{t_0}(R)$, for some $t \in (t_0, t^*]$. It has been observed that, by the strong comparison principle, this is possible if and only if $t = t^*$. Consequently, the choice of $u^*$ implies that $\mathcal{S}_{t_0}$ crosses $\{t^*\} \times \partial U_{t_0}(R)$ exactly in $(t^*, u^*)$. This proof is concluded by taking $\mathcal{C} = \cup_{t_0 < t^*} \mathcal{S}_{t_0}$. □

*Remark 10.4.3* We will see in the next chapter that the above proof can cover the case of a nonlinearity $f$ such that $\gamma_+ = +\infty$ (superlinear at $+\infty$).

# Chapter 11
# Superlinear Problems

This chapter deals with superlinear problems, i.e., nonlinear Dirichlet boundary value problems whose nonlinearity $f(u)$ is superlinear at $\infty$, namely

$$\lim_{u\to+\infty}\frac{f(u)}{u}=+\infty.$$

In this case an appropriate approach seems to be critical point theory. Actually, the mountain pass theorem or the linking theorem can be used to find solutions. We also show how to study superlinear problems by using the topological degree.

## 11.1 Using Min–Max Theorems

We will find solutions of problems with a superlinear nonlinearity by means of the min-max theorems proved in Sects. 5.3 and 5.5. For the reader's convenience we will first consider the model case

$$\begin{cases}-\Delta u=\lambda u+|u|^{p-1}u, & x\in\Omega\\ \quad u=0, & x\in\partial\Omega,\end{cases}\tag{11.1}$$

where $\lambda\geq 0$ is a parameter and $1<p<2^*-1$. Let us remark that $2^*$ is given by (7.3). The solutions of (11.1) are the critical points of

$$\mathcal{J}_\lambda(u)=\tfrac{1}{2}\|u\|^2-\tfrac{1}{2}\,\lambda\|u\|_2^2-\mathcal{H}(u),\quad u\in E=H_0^1(\Omega),$$

where

$$\mathcal{H}(u)=\tfrac{1}{p+1}\int|u|^{p+1}.$$

**Lemma 11.1.1** *If* $1<p<2^*-1$*, then* $\mathcal{J}_\lambda$ *satisfies the* $(PS)_c$ *for all* $c>0$.

*Proof* Let $u_n\in E$ be such that $\mathcal{J}_\lambda(u_n)\to c$ and $\mathcal{J}_\lambda'(u_n)\to 0$. From the former, resp. the latter multiplied by $u_n$, we get

$$\tfrac{1}{2}\|u_n\|^2-\tfrac{1}{2}\,\lambda\|u_n\|_2^2-\mathcal{H}(u_n)=c+\mathrm{o}(1),$$

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82,
DOI 10.1007/978-0-8176-8114-2_11, 

$$\|u_n\|^2 - \lambda\|u_n\|_2^2 - (\mathcal{H}'(u_n) \mid u_n) = \mathrm{o}(1).$$

From the first identity we infer $\|u_n\|^2 - \lambda\|u_n\|_2^2 = 2\mathcal{H}(u_n) + 2c + \mathrm{o}(1)$ and, inserting this into the second one, we deduce that

$$(\mathcal{H}'(u_n) \mid u_n) = 2\mathcal{H}(u_n) + 2c + \mathrm{o}(1). \tag{11.2}$$

Using the homogeneity of $\mathcal{H}$ ($\mathcal{H}'(u)(u) = (p+1)\mathcal{H}(u)$), we have

$$(p-1)\mathcal{H}(u_n) = 2c + \mathrm{o}(1).$$

Then $p > 1$ and the definition of $\mathcal{H}$ imply that $u_n$ is bounded in $L^p(\Omega)$ and this shows that $\|u_n\|_2^2 + \mathcal{H}(u_n)$ is bounded. Therefore $\|u_n\|$ is also bounded. The rest of the proof follows from Lemma 7.1.1. □

*Remark 11.1.2* The homogeneity of $\mathcal{H}$ can be substituted by the condition

$$\mathcal{H}(u) \le \theta(\mathcal{H}'(u) \mid u), \qquad \theta \in (0, \tfrac{1}{2}). \tag{11.3}$$

Actually, using (11.3) in (11.2) we get

$$\mathcal{H}(u_n) \le \theta(\mathcal{H}'(u_n) \mid u_n) = 2\theta\mathcal{H}(u_n) + 2\theta c + \mathrm{o}(1),$$

namely,

$$(1-2\theta)\mathcal{H}(u_n) \le 2\theta c + \mathrm{o}(1),$$

and the conclusion follows in the same way.

The geometrical properties of the functional $\mathcal{J}_\lambda$ depend on the value $\lambda$. Indeed, since the characteristic values of the operator $\lambda A$ are the decreasing sequence $\mu_j = \frac{1}{\lambda\lambda_j}$ ($j = 1, 2\dots$), applying Examples 5.3.1 and 5.5.1, we have the following result on the verification of the conditions $(J1)$–$(J4)$ introduced in Sects. 5.3 and 5.5.

**Lemma 11.1.3** *(i) If $\lambda < \lambda_1$, then $\mathcal{J}_\lambda$ satisfies $(J1)$ and $(J2)$.*
*(ii) If $\lambda_k \le \lambda < \lambda_{k+1}$, $k \ge 1$, then $\mathcal{J}_\lambda$ satisfies $(J3)$ and $(J4)$, with $V =$ span$\{\varphi_1, \dots, \varphi_k\}$.* □

The preceding lemmas allow us to apply the mountain pass theorem 5.3.6, resp. the min–max theorem 5.5.3, provided $\lambda < \lambda_1$, resp. $\lambda_k \le \lambda < \lambda_{k+1}$, yielding a nontrivial critical point of $\mathcal{J}_\lambda$ and hence a nontrivial solution of (11.1). Furthermore, if $\lambda < \lambda_1$, we can assert that the solution is positive in $\Omega$. Indeed, the same previous arguments work to prove the existence of a solution if we substitute the nonlinearity $|u|^{p-1}u$ with its positive part. In addition, we deduce by the maximum principle that the solution we find is positive.

Similar arguments apply to the problem

$$\begin{aligned} -\Delta u &= \lambda u + f(x,u), & x &\in \Omega, \\ u &= 0, & x &\in \partial\Omega, \end{aligned} \tag{11.4}$$

where $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies

*(i)* $f(x,0) = f_u(x,0) = 0$,
*(ii)* $|f(x,u)| \le a_1 + a_2|u|^p$ with $1 < p < 2^* - 1$,
*(iii)* $\exists\, \theta \in (0, \frac{1}{2})$ such that, letting $F(x,u) = \int_0^u f(x,s)ds$,

$$0 < F(x,u) \le \theta u f(x,u), \quad \forall\, x \in \Omega, \quad \forall u > 0. \tag{11.5}$$

In this more general case the preceding arguments require some modifications that we are going to outline.

As for the $(PS)$ condition, it suffices to point out that now one has that $\mathcal{H}(u) = \int F(x,u)\,dx$ as well as $(\mathcal{H}'(u) \mid u) = \int u f(x,u)\,dx$ and hence assumption $(iii)$ implies that (11.3) holds.

The verifications of $(J1)$ or $(J3)$ in Lemma 11.1.3 (see Example 5.3.1) do not depend on the homogeneity of the nonlinearity. On the other hand, from (11.5) it follows that $f(x,u)F^{-1}(x,u) \ge \theta^{-1}u^{-1}$ and hence, integrating,

$$|F(x,u)| \ge a|u|^{1/\theta}, \quad a > 0. \tag{11.6}$$

Using again (11.5), we deduce that $f$ is superlinear at infinity and allows us to repeat the arguments carried out in Example 5.5.1 proving that $(J2)$ or $(J4)$ holds.

In conclusion, we can state the following result.

**Theorem 11.1.4** *If $f$ satisfies $(i)$, $(ii)$ and $(iii)$, then (11.4) has a nontrivial solution. Moreover, if $\lambda < \lambda_1$, (11.4) has a positive solution.* □

*Remark 11.1.5* Condition $(iii)$ can be further weakened by requiring that it hold only for all $|u| \gg 1$. The proof requires some minor changes that are left to the reader.

*Remark 11.1.6* Exercises 40 and 41 show that, in general, positive solutions given by the preceding theorem cannot be obtained by sub- and super-solutions.

*Remark 11.1.7* In general, (11.4) *has no nontrivial solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ if $\lambda \le 0$, $p \ge 2^* - 1$ and $N > 2$.* This can be derived as a consequence of an integral identity for the case $f(x,u) = f(u)$, due to Pohozaev, which states that *any solution of* (11.4) *verifies*

$$N\int F(u) - \frac{N-2}{2}\int u f(u) + \lambda \int u^2 = \tfrac{1}{2}\int_{\partial\Omega} u_\nu^2 (x\cdot\nu)d\sigma, \tag{11.7}$$

*where $\nu$ is the unit outer normal on $\partial\Omega$ and $u_\nu = \frac{\partial u}{\partial \nu}$.* Roughly, (11.7) follows by multiplying (11.4) by $x \cdot \nabla u$ to deduce that

$$\begin{aligned} f(u)\,x\cdot\nabla u &= -\Delta u\,(x\cdot\nabla u) \\ &= -\mathrm{div}\,((x\cdot\nabla u)\,\nabla u) + \frac{1}{2}\mathrm{div}\,(x|\nabla u|^2) - \frac{N-2}{2}|\nabla u|^2, \end{aligned}$$

which, integrating in $\Omega$ and using the divergence theorem, implies

$$N\int F(u) = -\int_{\partial\Omega}(x\cdot\nabla u)\,u_\nu + \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(x\cdot\nabla u) - \frac{N-2}{2}\int|\nabla u|^2.$$

Finally, (11.7) is obtained by observing that $\nabla u = u_\nu\,\nu$ (since $u = 0$ on $\partial\Omega$) and taking into account that, by choosing $u$ as a test function in (11.4),

$$\int|\nabla u|^2 = \int f(u)u.$$

When $f(u) = |u|^{p-1}u$, the left-hand side of (11.7) becomes

$$\left(\frac{N}{p+1} - \frac{N-2}{2}\right)\int|u|^{p+1} + \lambda\int u^2.$$

If, in addition, $\lambda \le 0$ and the set $\Omega$ is *star-shaped*, i.e., such that $x\cdot\nu > 0$ on $\partial\Omega$, we infer from (11.7) that

$$\frac{N}{p+1}\int|u|^{p+1} > \frac{N-2}{2}\int|u|^{p+1}.$$

Therefore, if (11.4) has a nontrivial solution, then $p+1 < 2N/(N-2) = 2^*$.

*Remark 11.1.8* In contrast to the previous discussion, if we consider the following linear perturbation of problem (11.4):

$$\begin{aligned} -\Delta u &= \lambda u + |u|^{2^*-2}u, \quad & x\in\Omega \\ u &= 0, & x\in\partial\Omega, \end{aligned} \tag{11.8}$$

and $N \ge 4$, then (11.8) has a positive solution whenever $0 < \lambda < \lambda_1$. In the case $N = 3$ there exists $\lambda^* \ge 0$ such that (11.8) has a positive solution whenever $\lambda^* < \lambda < \lambda_1$. Moreover, if $\Omega$ is a ball, necessarily $\lambda^* > 0$.

These and other results dealing with (11.8), including existence of solutions for $\lambda > \lambda_1$, are out of the scope of this book. For an exposition, we refer, e.g., to [37].

## 11.2 Superlinear Ambrosetti–Prodi Problem

In this section we study a superlinear version of the Ambrosetti and Prodi problem. Specifically, we consider the boundary value problem

$$\begin{aligned} -\Delta u &= f(x,u) + t\varphi, \quad & x\in\Omega, \\ u &= 0, & x\in\partial\Omega, \end{aligned} \tag{11.9$_t$}$$

where $\varphi \in L^\infty(\Omega)$ is a positive function, and $f$ is a continuous function such that

$$\lim_{s\to-\infty}\frac{f(x,s)}{s} = \gamma_-(x) \le \lambda_1 - \varepsilon, \quad \text{uniformly in } x\in\Omega, \tag{11.10}$$

for some $\varepsilon > 0$. In addition, we suppose that there exists $h(x) \in L^{\infty}(\Omega)$ and $1 < p < 2^* - 1$ such that

$$\lim_{s \to +\infty} \frac{f(x,s)}{s^p} = h(x) > c > 0, \quad \text{uniformly in } x \in \Omega. \tag{11.11}$$

Notice that this hypothesis implies that $f$ is superlinear at $+\infty$ and satisfies the subcritical condition $(ii)$ of the previous section.

**Theorem 11.2.1** *Let $\varphi \in L^{\infty}(\Omega)$ be a positive function and let $f$ be a continuous function satisfying (11.10) and (11.11). Then $t^*$, the supremum of all $t \in \mathbb{R}$ such that problem $(11.9_t)$ admits a solution, is finite and there exists a continuum $\mathcal{C}$ in $\Sigma \equiv \{(t,u) \in \mathbb{R} \times C_0^1(\overline{\Omega}) :\ u \text{ solution of } (11.9_t)\}$ satisfying that*

1. $(-\infty, t^*] \subset Proj_{\mathbb{R}}\, \mathcal{C}$.
2. *For every $t \in (-\infty, t^*)$, the $t$-slice $\mathcal{C}_t = \{u \in C_0^1(\overline{\Omega}) : (t,u) \in \mathcal{C}\}$ contains two distinct solutions of $(11.9_t)$.* □

*Remark 11.2.2* Similarly to Remark 10.4.2, we obtain as a corollary that problem $(11.9_t)$ has, at least, two (respectively, one, zero) solutions for $t < t^*$ (respectively, $t \le t^*, t > t^*$.).

The proof is essentially equal to the one of Theorem 10.4.1. The only change is the estimate given by Lemma 10.2.1. In this case, an easy extension of the result by Gidas and Spruck in [57] gives the following result.

**Lemma 11.2.3** *Let $\varphi \in L^{\infty}(\Omega)$ be a positive function. Suppose that $f$ satisfies (11.10) and (11.11). Then the solutions of $(11.9_t)$ are uniformly bounded in compact sets of $t$, i.e., for every compact interval $\Gamma \subset \mathbb{R}$, there exists $c \in \mathbb{R}$ such that every solution $u$ of $(11.9_t)$ with $t \in \Gamma$ satisfies*

$$\|u\|_{C^1} \le c.$$

*Proof* By bootstrap arguments, it is sufficient to prove the existence of an a priori estimate for the norm in $L^{\infty}(\Omega)$ of the solutions of $(11.9_t)$ with $t$ in a given compact interval $\Gamma$. The proof is divided into two steps:

*Step* 1. There exists a positive constant $c$ such that

$$u(x) > -c,\ x \in \Omega,$$

for every solution $u$ of $(P_t)$ with $t \in \Gamma$.

*Step* 2. There exists a positive constant $C$ such that

$$u(x) \le C,\ x \in \Omega,$$

for every solution $u$ of $(P_t)$ with $t \in \Gamma$.

*Proof of Step* 1 In order to prove this a priori bound, we observe that, taking $u^- \equiv \min\{u, 0\}$ as a test function in the equation satisfied by $u$, and by using hypothesis

(11.10), we get a uniform bound in the $H_0^1(\Omega)$-norm of $u^-$. For each $k \in \mathbb{R}$, the function $G_k$ is given by

$$G_k(s) = \begin{cases} s+k, & \text{if } s \le -k, \\ 0, & \text{if } -k < s \le k, \\ s-k, & \text{if } k < s. \end{cases}$$

Taking $v = G_k(u^-)$ as a test function in the equation satisfied by $u$, we obtain that

$$\int |\nabla G_k(u^-)|^2 = \int_{\Omega_k} [f(x,u^-) + t\varphi] G_k(u^-),$$

where $\Omega_k \equiv \{x \in \Omega : \ u(x) < -k\}$. From (11.10), there exists a positive constant $C$ such that

$$f(x,s) + t\varphi \ge Cs, \quad \forall s \le -k, \quad \forall t \in \Gamma.$$

We deduce from above that

$$\int |\nabla G_k(u^-)|^2 \le C \int_{\Omega_k} |u^-||G_k(u^-)|.$$

Using now the Sobolev inequality, we get

$$\|G_k(u^-)\|_{2^*}^2 \le C_1 \int |\nabla G_k(u^-)|^2 \le C_2 \int_{\Omega_k} |u^-||G_k(u^-)|.$$

Moreover, if $r > 2N/(N+2)$, by the Hölder inequality, we infer

$$\int_{\Omega_k} |u^-||G_k(u^-)| \le \|u^-\|_r \|G_k(u^-)\|_{2^*} |\Omega_k|^{(1-1/r-1/2^*)}.$$

Hence,

$$\|G_k(u^-)\|_{2^*}^2 \le C_2 \|u^-\|_r \|G_k(u^-)\|_{2^*} |\Omega_k|^{(1-1/r-1/2^*)}.$$

Notice now that for every $h \ge k$, $|G_k(u^-)| \ge h - k$ in $\Omega_h$, which implies that

$$(h-k)|\Omega_h|^{1/2^*} \le C_2 \|u^-\|_r |\Omega_k|^{(1-1/r-1/2^*)},$$

or equivalently that

$$|\Omega_h| \le \frac{C_2 \|u^-\|_r^{2^*} |\Omega_k|^{(2^*-1-2^*/r)}}{(h-k)^{2^*}}.$$

The following lemma of real analysis can be found in [83, Lemme 4.1, p. 19].

**Lemma 11.2.4** *Assume that $k_1 \ge 0$, $C, \alpha, \beta > 0$ and that $\Psi(h)$ is a non-increasing and non-negative function satisfying*

$$\Psi(h) \le \frac{C}{(h-k)^\alpha} \Psi(k)^\beta, \quad \forall h > k \ge k_1.$$

*If $\beta > 1$, then $\Psi(h_0) = 0$, with $h_0 = k_1 + (C\Psi(k_1)^{\beta-1} 2^{\alpha\beta/(\beta-1)})^{1/\alpha}$.* □

Applying the previous lemma with $\Psi(h) := |\Omega_h|$, we deduce the existence of a positive constant $h_0$ such that $|\Omega_{h_0}| = 0$ and hence that $\|u^-\|_\infty \le$ const., for every solution $u$ of $(P_t)$ with $t \in \Gamma$. Therefore, Step 1 has been proved.

*Proof of Step 2* Consider the constant $c$ obtained in the first step. It suffices to show the existence of $\widetilde{c} \in \mathbb{R}^+$ such that $v(x) := u(x) + c \le \widetilde{c}$, for every $x \in \Omega$. Since $v = u + c > 0$ satisfies

$$\begin{aligned} -\Delta v &= \widetilde{f}(x,v) + t\varphi, & x \in \Omega,\\ v &= c, & x \in \partial\Omega, \end{aligned}$$

with $\widetilde{f}(x,s) := f(x, s-c)$, we follow the outline of [57] where the case $c = 0$ is studied. Arguing by contradiction, assume that there exist positive solutions $v_n \in C^1(\overline{\Omega})$ of the above problem with $\lambda_n \in \Gamma$ and points $P_n \in \Omega$ such that:

$$c < M_n = \max_{\overline{\Omega}} v_n = v_n(P_n) \to +\infty.$$

Then, up to a subsequence, we may assume

$$\lambda_n \to \lambda, \quad P_n \to P \in \overline{\Omega}.$$

Two cases can occur: either $P \in \Omega$ or $P \in \partial\Omega$. In both cases, we will obtain a contradiction.

Indeed, in the first case, i.e., $P \in \Omega$, let $d = \operatorname{dist}(P, \partial\Omega)/2 > 0$, $\mu_n = M_n^{\frac{1-p}{2}}$ and

$$w_n(y) = \mu_n^{\frac{2}{p-1}} v_n(P_n + \mu_n y), \tag{11.12}$$

for every $y$ in the ball $B_{\frac{d}{\mu_n}}(0)$ of center 0 and radius $\frac{d}{\mu_n}$. Observe that $\mu_n \to 0$,

$$\sup_{B_{\frac{d}{\mu_n}}(0)} w_n = w_n(0) = 1,$$

and $w_n$ satisfies

$$-\Delta w_n(y) = g_n(y), \quad y \in B_{\frac{d}{\mu_n}}(0), \tag{11.13}$$

where

$$g_n(y) = \mu_n^{\frac{2p}{p-1}} f\left(\lambda_n, \mu_n y + P_n, \mu_n^{\frac{-2}{p-1}} v_n(y)\right).$$

By $L^p$-theory, we get that $v_n \in W^{2,s}(B_{\frac{d}{\mu_n}}(0))$, for every $s > 1$. In addition, if we fix $R > 0$ and let $n_0$ be a positive integer such that $R < d/\mu_n$ for every $n \ge n_0$, we obtain for every $R' \in (R, d/\mu_n)$ that

$$\|w_n\|_{W^{2,s}(B_R(0))} \le C\left(\|w_n\|_{L^s(B_{R'}(0))} + \|g_n\|_{L^s(B_{R'}(0))}\right),$$

where $C$ is a positive constant depending only on $N$, $p$, $\alpha$, $\beta$ and $R'$.

By (11.11), the right-hand side of this equation satisfies

$$\lim_{n\to\infty}\left|\mu_n^{\frac{2p}{p-1}} f\left(\lambda_n, \mu_n y + P_n, \mu_n^{\frac{-2}{p-1}} v_n(y)\right) - h(\mu_n y + P_n) v_n(y)^p\right| = 0. \quad (11.14)$$

Taking into account that

$$\|w_n\|_{L^s(B_{R'}(0))} + \|g_n\|_{L^s(B_{R'}(0))} \le C_1 = C_1(R', \lambda_n), \ \forall n \ge n_0,$$

we deduce a uniform bound for $\|w_n\|_{W^{2,s}(B_R(0))}$ for every $n \ge n_0$. Choosing $s$ large enough, we obtain from Morrey's theorem (see Theorem A.4.3) that $\|w_n\|_{C^{1,\beta}(\overline{B_R(0)})}$ is uniformly bounded. Therefore we can apply the Ascoli–Arzelà theorem and deduce the existence of a function $w \in C(\overline{B_R(0)})$ such that, up to a subsequence, $w_n \to w$ in $C(\overline{B_R(0)})$ and $h(\mu_n y + P_n) \to \nu$, for some $\nu > 0$. Necessarily, we have that $w(0) = 1$ and, using again (11.14),

$$-\Delta w = \nu w^p, \quad y \in B_R(0).$$

Then, by regularity, for $\tau \in (0, 1)$, $w \in C^{1,\tau}(\overline{B_R(0)})$. From the arbitrariness of $R > 0$ we deduce that $w$ is defined in $\mathbb{R}^N$ and it is a solution of

$$-\Delta w(y) = w^p(y), \quad y \in \mathbb{R}^N.$$

By Theorem 1.2 in [57], $w \equiv 0$, contradicting that $w(0) = 1$.

In the second case, $P \in \partial\Omega$ and since $\partial\Omega$ is smooth, we can suppose that near $P$ the boundary of $\Omega$ is contained in the hyperplane $x_N = 0$ and that a neighborhood of $P$ in $\Omega$ is contained in the set $\{x \in \mathbb{R}^N : x_N > 0\}$. We set $d_n = \mathrm{dist}(P_n, \partial\Omega) = P_n \cdot e_n$, ($e_n = (0, \dots, 0, 1)$), and we observe that the function $w_n$ given by (11.12) is well defined in $\Omega_n \equiv B_{\frac{\delta}{\mu_n}}(0) \cap \{y_N > -d_n/\mu_n\}$, for some $\delta > 0$. Moreover, it satisfies (11.13) in $\Omega_n$. By $L^p$-theory up to the boundary (see Theorem 1.2.11-1) and Morrey's theorem, we deduce again that $|\nabla w_n|$ is uniformly bounded in $\Omega_n$. Consequently,

$$1 = \left|w_n(0) - w_n\left(-\frac{d_n}{\mu_n} e_n\right)\right| \le C\frac{d_n}{\mu_n},$$

i.e., $d_n/\mu_n$ is away from zero. If, for a subsequence, $d_n/\mu_n \to \infty$, we can apply similar arguments to those of the first case to reach again a contradiction. On the other hand, if $d_n/\mu_n$ is bounded from above, we assume, passing to a subsequence if necessary, that $d_n/\mu_n \to s > 0$. Since $w_n$ satisfies (11.13) in $\Omega_n$, again by $L^p$-theory, for every $R, \varepsilon > 0$, we get a uniform bound of $w_n$ in $C^{1,\tau}(B_R(0) \cap \{y_N > -s + \varepsilon\})$ for $n$ large enough. Therefore, we obtain that, up to a subsequence, $w_n \to w$ in $C^1(B_R(0) \cap \{y_N > -s + \varepsilon\})$, $h(\mu_n y + P_n) \to \nu$, for some $\nu > 0$, and using that $R$ and $\varepsilon$ are arbitrary, $w$ is a solution of

$$\begin{cases} -\Delta w = \nu w^p, & \{y_N > -s\}, \\ w(y) = 0, & \{y_N = -s\}. \end{cases}$$

Theorem 1.3 in [57] implies then that $w \equiv 0$, a contradiction with $w(0) = 1$. □

It is worthwhile to observe that the above theorem implies also the existence of positive solution of (11.4) provided that $\lambda < \lambda_1$.

**Corollary 11.2.5** *If $f$ is a continuous function in $\overline{\Omega} \times [0, +\infty)$ satisfying $(i)$ of the previous section and (11.11), then problem (11.4) has a positive solution provided that $\lambda < \lambda_1$.*

*Remark 11.2.6* Compare this result with the assertion proved in Theorem 11.1.4 for $\lambda < \lambda_1$.

*Proof* To prove the corollary, we extend, as usual when we look for positive solutions, the nonlinearity to all $\overline{\Omega} \times \mathbb{R}$ by setting $f(x, u) = 0$ for $u < 0$. Choosing $\varphi = \varphi_1$, we embed the problem (11.4) into the one-parameter family of problems $(11.9_t)$. Applying Theorem 11.2.1, we deduce the existence of, at least, two solutions of $(11.9_t)$ for $t \leq t^*$. The proof will be concluded if we show that $t^* > 0$ because this implies that $(11.9_0)$, i.e. (11.4) has two solutions: one is the zero one and the other the positive solution that we are looking for. To prove that $t^* > 0$, it suffices to observe that, by condition $(i)$ and for $t > 0$, $\delta\varphi_1$ is a super-solution of $(11.9_t)$ provided that $\delta > 0$ is sufficient small. □

# Chapter 12
# Quasilinear Problems

In this chapter we consider a class of quasilinear elliptic problems. In order to handle this case an improvement of the mountain pass theorem is needed because the Euler functional fails to be $C^1$. This critical point result is discussed in Sect. 12.2 and is applied to boundary value problems in Sect. 12.3. A nonvariational equation is also considered in Sect. 12.4, where we apply the global bifurcation theorem (see Theorem 4.4.1).

## 12.1 First Results

We study quasilinear Dirichlet problems in a bounded open set $\Omega \subset \mathbb{R}^N$, where $N \geq 3$ (the case $N = 2$ is left to the reader (see Exercise 49)). Specifically, we replace the linear operator $\Delta$ by a quasilinear operator, namely we consider here operators like $Qu = -\mathrm{div}\,(a(x,u)\nabla u) + g(x,u)|\nabla u|^2$, where $a(x,s)$ and $g(x,s)$ are continuous functions in $\Omega \times \mathbb{R}$ satisfying

$$\alpha \leq a(x,s) \leq \beta, \quad \forall x \in \Omega, \quad \forall s \in \mathbb{R}, \tag{12.1}$$

for positive constants $\alpha$, $\beta$, and

$$g(x,s)s \geq 0, \quad \forall x \in \Omega, \quad \forall s \in \mathbb{R}. \tag{12.2}$$

Observe that $Q$ contains a lower order term with quadratic growth with respect to the gradient. From the works of Boccardo et al. [33–35] this kind of quasilinear operator has been extensively studied, especially if the right-hand side is linear. In particular, among other results, these authors have proved the following one.

**Theorem 12.1.1** *Assume that $a(x,s)$ and $g(x,s)$ are continuous functions in $\Omega \times \mathbb{R}$ satisfying (12.1) and (12.2). If $h(x) \in L^q(\Omega)$ with $q > \frac{N}{2}$, then the problem*

$$\begin{aligned} -div\,(a(x,u)\nabla u) + g(x,u)|\nabla u|^2 &= h(x), \quad x \in \Omega \\ u &= 0, \qquad x \in \partial\Omega \end{aligned}$$

*has a solution $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$.* □

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82,
DOI 10.1007/978-0-8176-8114-2_12, 

When $a$ and $g$ are independent of $x \in \Omega$, i.e., $a(x,s) = \alpha(s)$ and $g(x,s) = g(s)$, the uniqueness of solution is a consequence of the following principle [27].

**Theorem 12.1.2** *Given a continuous function* $g : [0,+\infty) \longrightarrow [0,+\infty)$ *and* $q > \frac{N}{2}$, *let* $0 \le h_1, h_2 \in L^q(\Omega)$ *be functions such that*

$$h_1(x) \ge h_2(x), \quad \forall x \in \Omega.$$

*If* $0 \le u, v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *verify*

$$-\mathrm{div}\,(\alpha(u)\nabla u) + g(u)|\nabla u|^2 = h_1, \quad x \in \Omega$$

*and*

$$-\mathrm{div}\,(\alpha(v)\nabla v) + g(v)|\nabla v|^2 = h_2, \quad x \in \Omega,$$

*then* $u \le v$.

*Proof* Define $\gamma(s) = \int_0^s g(t)\,dt$ and $P(s) = \int_0^s \alpha(t)e^{-\gamma(t)}\,dt$, for every $s > 0$. Taking

$$e^{-\gamma(u)}\,[P(u) - P(v)]^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

as a test function in the equation satisfied by $u$, we obtain

$$\int \nabla u \cdot \nabla \left[e^{-\gamma(u)}\,[P(u) - P(v)]^+\right] + \int g(u)|\nabla u|^2 e^{-\gamma(u)}\,[P(u) - P(v)]^+$$
$$= \int h_1 e^{-\gamma(u)}\,[P(u) - P(v)]^+ . \qquad (12.3)$$

Using that $\gamma' = g$, we have

$$\nabla \left[e^{-\gamma(u)}\,[P(u) - P(v)]^+\right] = e^{-\gamma(u)}\nabla\,[P(u) - P(v)]^+$$
$$- g(u)\nabla u\, e^{-\gamma(u)}\,[P(u) - P(v)]^+ .$$

Hence (12.3) means

$$\int e^{-\gamma(u)}\nabla u \cdot \nabla\,[P(u) - P(v)]^+ = \int h_1 e^{-\gamma(u)}\,[P(u) - P(v)]^+ .$$

Since $\nabla P(u) = P'(u)\nabla u = e^{-\gamma(u)}\nabla u$, we can rewrite the above equality as

$$\int \nabla P(u) \cdot \nabla\,[P(u) - P(v)]^+ = \int h_1 e^{-\gamma(u)}\,[P(u) - P(v)]^+ . \qquad (12.4)$$

Similarly, taking now $e^{-\gamma(v)}\,[P(u) - P(v)]^+$ as a test function in the equation of $v$, we deduce that

$$\int \nabla P(v) \cdot \nabla\,[P(u) - P(v)]^+ = \int h_2 e^{-\gamma(v)}\,[P(u) - P(v)]^+ .$$

Subtracting this from (12.4) we get from the non-negativeness of $h_1 - h_2$

$$\int |\nabla\,[P(u) - P(v)]^+|^2 \le \int (h_1 e^{-\gamma(u)} - h_2 e^{-\gamma(v)})\,[P(u) - P(v)]^+ \le 0,$$

i.e., $P(u) \le P(v)$ or equivalently $u \le v$. □

By Theorems 12.1.1 and 12.1.2, if $2^*$ is given by (7.3), given $p \in (1, 2^* - 1)$, we can consider the operator $K^g : \mathbb{R} \times H_0^1(\Omega) \to H_0^1(\Omega)$ by defining, for every $\lambda \in \mathbb{R}$ and for every $w \in H_0^1(\Omega)$, $K^g(\lambda, w)$ as the unique solution $u$ in $H_0^1(\Omega)$ of the problem

$$\begin{aligned} -\Delta u + g(u)|\nabla u|^2 &= \lambda^+ w^+(x)^p + h(x), \quad && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

In the following result we state without proof the compactness property of $K^g(\lambda, w)$. For details see [22].

**Proposition 12.1.3** *If the sequences $t_n \in [0, 1]$ and $\lambda_n > 0$ are convergent, respectively, to $t^*$ and $\lambda$, and $w_n$ is $H_0^1(\Omega)$-weakly convergent to $w$, then the sequence of the unique solution $u_n \in H_0^1(\Omega)$ of*

$$\begin{aligned} -\Delta u_n + t_n g(u_n)|\nabla u_n|^2 &= \lambda_n w_n^+(x)^p + h(x), \quad && x \in \Omega, \\ u_n &= 0, && x \in \partial\Omega, \end{aligned}$$

*is strongly convergent in $H_0^1(\Omega)$ to the solution $u$ of*

$$\begin{aligned} -\Delta u + t^* g(u)|\nabla u|^2 &= \lambda w^+(x)^p + h(x), \quad && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

□

## 12.2 Mountain Pass Theorem for Nondifferentiable Functionals and Applications

The study of quasilinear problems associated to the operator $Q$ of the preceding section with a superlinear right-hand side requires the extension of the mountain pass Theorem 11.1.4 to cover the case of functionals which are not differentiable in all directions.

The proof of the classical mountain pass theorem given in Sect. 5.3 was based on the deformation lemma (Lemma 5.3.2). A different approach based on the Ekeland variational principle (Theorem 5.4.2) can be found in [29, 48, 69]. In this section, we prove the required extension by following the latter strategy. All the ideas used here are close to those in [19].

**Theorem 12.2.1** *Let $X$ be a Banach space with a norm $\|\cdot\|_X$ and $Y \subset X$ a subspace, which is itself a Banach space endowed with a different norm $\|\cdot\|_Y$. Assume that $\mathcal{J} : X \longrightarrow \mathbb{R}$ is a functional on $X$ such that $\mathcal{J}_{|Y}$ is continuous in $(Y, \|\cdot\|_X + \|\cdot\|_Y)$ and satisfies the following hypotheses:*

*(a) $\mathcal{J}$ has a directional derivative $\langle \mathcal{J}'(u), v\rangle$ at each $u \in X$ through any direction $v \in Y$.*
*(b) For fixed $u \in X$, the function $\langle \mathcal{J}'(u), v\rangle$ is linear in $v \in Y$, and, for fixed $v \in Y$, the function $\langle \mathcal{J}'(u), v\rangle$ is continuous in $u \in X$.*

*Assume that for* $e \in Y$,

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)) > c_1 = \max\{\mathcal{J}(0), \mathcal{J}(e)\} \tag{12.5}$$

*with* $\Gamma$ *the set of the continuous paths* $\gamma : [0,1] \longrightarrow (Y, \|\cdot\|_X + \|\cdot\|_Y)$ *such that* $\gamma(0) = 0$ *and* $\gamma(1) = e$. *Suppose, in addition, that* $\mathcal{J}$ *satisfies the condition*

*(C) Any sequence* $\{u_n\}$ *in* $Y$ *satisfying for some* $\{K_n\} \subset (0, +\infty)$ *and* $\{\varepsilon_n\} \longrightarrow 0$ *the conditions*

$$\{\mathcal{J}(u_n)\} \text{ is bounded,} \tag{12.6}$$

$$\|u_n\|_Y \le 2K_n \ \ \forall n \in \mathbb{N}, \tag{12.7}$$

$$|\langle \mathcal{J}'(u_n), v\rangle| \le \varepsilon_n \left[\frac{\|v\|_Y}{K_n} + \|v\|_X\right] \ \ \forall v \in Y, \tag{12.8}$$

*possesses a convergent subsequence in* $X$.

*Then* $c$ *is a critical value of* $\mathcal{J}$, *i.e., there exists a (nonzero) point* $u \in Y$ *such that* $\mathcal{J}(u) = c$ *and which is a critical point of* $\mathcal{J}$: $\langle \mathcal{J}'(u), v\rangle = 0$, $\forall v \in Y$.

*Proof* Consider the functional $\mathcal{G}$ defined on $\Gamma$ by setting

$$\mathcal{G}(\gamma) = \max_{t\in[0,1]} \mathcal{J}(\gamma(t)), \quad \forall \gamma \in \Gamma.$$

By (12.5) we observe that $\mathcal{G}$ is bounded from below with infimum $c$. Let us consider $\varepsilon_n := \frac{c-c_1}{n}$ and a sequence $\{\gamma_n\}$ of minimizing paths in $\Gamma$ satisfying

$$c \le \mathcal{G}(\gamma_n) \le c + \frac{\varepsilon_n}{2}.$$

Let us denote $M_n := \max_{t\in[0,1]} \|\gamma_n(t)\|_Y \ge \|e\|_Y$. For each fixed $n \in \mathbb{N}$, we consider the distance $\mathrm{d}_n$ in $\Gamma$ given by $\mathrm{d}_n(\gamma, \widetilde{\gamma}) = \max_{t\in[0,1]} \frac{\|\gamma(t)-\widetilde{\gamma}(t)\|_Y}{M_n} + \|\gamma(t) - \widetilde{\gamma}(t)\|_X$, for $\gamma, \widetilde{\gamma} \in \Gamma$. Equipped with this, $\Gamma$ is a complete metric space and $\mathcal{G}$ is also a lower semicontinuous functional. Hence, applying the Ekeland variational principle (see Theorem 5.4.2), we deduce that there exists $\overline{\gamma}_n \in \Gamma$ satisfying

$$c \le \mathcal{G}(\overline{\gamma}_n) \le \mathcal{G}(\gamma_n) \le c + \frac{\varepsilon_n}{2},$$

$$\mathrm{d}_n(\overline{\gamma}_n, \gamma_n) = \max_{t\in[0,1]} \frac{\|\gamma_n(t) - \overline{\gamma}_n(t)\|_Y}{M_n} + \|\gamma_n(t) - \overline{\gamma}_n(t)\|_X \le \sqrt{\varepsilon_n}, \tag{12.9}$$

and

$$\mathcal{G}(\overline{\gamma}_n) < \mathcal{G}(\vartheta) + \sqrt{\varepsilon_n}\,\mathrm{d}_n(\overline{\gamma}_\varepsilon, \vartheta), \quad \forall \vartheta \in \Gamma \setminus \{\overline{\gamma}_n\}. \tag{12.10}$$

Now we show that there exists $t_n \in \mathcal{T} := \{t \in [0,1] \ : \ c - \sqrt{\varepsilon_n} \le \mathcal{J}(\overline{\gamma}_n(t))\}$ such that $u_n := \overline{\gamma}_n(t_n)$ satisfies

$$|\langle \mathcal{J}'(u_n), v\rangle| \le \sqrt{\varepsilon_n}\left[\frac{\|v\|_Y}{M_n} + \|v\|_X\right], \quad \forall v \in Y. \tag{12.11}$$

Indeed, if by contradiction we assume that for every $t \in \mathcal{T}$ there exists $v_t \in Y$ such that $\frac{\|v_t\|_Y}{M_n} + \|v_t\|_X = 1$ and $\langle \mathcal{J}'(\overline{\gamma}_n(t)), v_t\rangle < -\sqrt{\varepsilon_n}$, then, by hypothesis *b)*, there exist $\delta_t > 0$ and an open neighborhood $B_t$ of $t$ in $[0,1]$ such that

$$\langle \mathcal{J}'(\overline{\gamma}_n(s) + u), v_t\rangle < -\sqrt{\varepsilon_n} \tag{12.12}$$

for every $s \in B_t$ and $u \in X$ such that $\|u\|_X < \delta_t$. Since $\mathcal{T}$ is compact, there exists a finite family of neighborhoods $B_{t_1}, B_{t_2}, \dots, B_{t_k}$ such that $\mathcal{T} \subset \cup_{j=1}^k B_{t_j}$. Take $\delta = \min\{\delta_{t_1}, \delta_{t_2}, \dots, \delta_{t_k}\}$ and choose functions $\psi, \psi_j \in C([0,1],[0,1])$ satisfying

$$\psi(s) = \begin{cases} 1, & \text{if } c \le \mathcal{J}(\overline{\gamma}_n(s)), \\ 0, & \text{if } \mathcal{J}(\overline{\gamma}_n(s)) \le c - \varepsilon_n \end{cases}$$

and

$$\psi_j(s) = \begin{cases} \dfrac{\operatorname{dist}(s,[0,1]-B_{t_j})}{\displaystyle\sum_{i=1}^k \operatorname{dist}(s,[0,1]-B_{t_i})}, & \text{if } s \in \cup_{i=1}^k B_{t_i}, \\ \\ 0, & \text{if } s \in [0,1] - \cup_{i=1}^k B_{t_i}. \end{cases}$$

It is easy to check that $\gamma^* := \overline{\gamma}_n + \delta\psi\sum_{j=1}^k \psi_j v_{t_j} \in \Gamma$. Note also that for every $s \in [0,1] - \mathcal{T}$, we have $\gamma^*(s) = \overline{\gamma}_n(s)$ and thus $\mathcal{J}(\gamma^*(s)) = \mathcal{J}(\overline{\gamma}_n(s)) < c - \varepsilon_n$. On the other hand, if $s \in \mathcal{T}$, hypothesis *(a)* and the mean value theorem imply the existence of $\tau \in (0,1)$ such that

$$\begin{aligned}
\mathcal{J}(\gamma^*(s)) - \mathcal{J}(\overline{\gamma}_n(s)) &= \langle \mathcal{J}'(\overline{\gamma}_n(s) + \tau\delta\psi(s)\sum_{j=1}^k \psi_j(s)v_{t_j}), \delta\psi(s)\sum_{j=1}^k \psi_j(s)v_{t_j}\rangle \\
(\text{by } (b)) &= \delta\psi(s)\sum_{j=1}^k \psi_j(s)\langle \mathcal{J}'(\overline{\gamma}_n(s) + \tau\delta\psi(s)\sum_{j=1}^k \psi_j(s)v_{t_j}), v_{t_j}\rangle \\
(\text{by } (12.12)) &\le -\delta\psi(s)\sqrt{\varepsilon_n}\sum_{j=1}^k \psi_j(s) \\
&= -\delta\sqrt{\varepsilon_n}\psi(s).
\end{aligned}$$

Consequently, if $s$ is the point in $[0,1]$ in which $\mathcal{J} \circ \gamma^*$ attains its maximum, i.e., $\mathcal{J}(\gamma^*(s)) = \mathcal{G}(\gamma^*) \ge c$, we deduce necessarily that $\psi(s) = 1$, $s \in \mathcal{T}$, and

$$\begin{aligned}
\mathcal{G}(\gamma^*) = \mathcal{J}(\gamma^*(s)) &\le \mathcal{J}(\overline{\gamma}_n(s)) - \delta\sqrt{\varepsilon_n} \le \mathcal{G}(\overline{\gamma}_n) - \delta\sqrt{\varepsilon_n} \\
&\le \mathcal{G}(\overline{\gamma}_n) - \sqrt{\varepsilon}\mathrm{d}_n(\gamma^*, \overline{\gamma}_n).
\end{aligned}$$

This contradicts (12.10) and proves the existence of $u_n \in \overline{\gamma}_n(\mathcal{T})$ such that $c - \sqrt{\varepsilon_n} \leq \mathcal{J}(u_n) \leq c + \frac{\varepsilon_n}{2}$ and (12.11) holds.

In addition, by (12.9),

$$\|u_n\|_Y = \|\overline{\gamma}_n(t_n)\|_Y \leq \|\overline{\gamma}_n(t_n) - \gamma_n(t_n)\|_Y + \|\gamma_n(t_n)\|_Y \leq (1 + M_n)\sqrt{\varepsilon_n}.$$

This means that the sequence $\{u_n\}$ satisfies (12.6)–(12.8) and therefore we conclude the proof by using the compactness condition $(C)$. □

*Remark 12.2.2* A similar result was proved in [39] using a nonsmooth critical point theorem for continuous functionals due to Corvellec et al. [42] together with a result of Boccardo et al. [35]. Moreover, the existence of nontrivial critical points for one-dimensional general functionals is proved in [63].

## 12.3 Application to Quasilinear Variational Problems

We apply the previous theorem to study the critical points of the functional

$$\mathcal{J}(u) = \frac{1}{2}\int a(x,u)|\nabla u|^2 - \int F(x,u), \quad u \in X := H_0^1(\Omega),$$

where $\Omega$ is an open set in $\mathbb{R}^N$ and, for positive constants $\alpha$, $\beta$ and $\gamma$, $a(x,s)$ is a Carathéodory function satisfying (12.1) and with Carathéodory derivative $a_s'(x,s)$ with respect to the variable $s$ such that, for some $\gamma > 0$,

$$|a_s'(x,s)| \leq \gamma, \tag{12.13}$$

for every $s \in \mathbb{R}$ and a.e. $x \in \Omega$. We also assume that $F(x,u) = \int_0^u f(x,s)ds$ where $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies the conditions (like in Theorem 11.1.4)

(i) $f(x,0) = f_u(x,0) = 0$,
(ii) $|f(x,u)| \leq C_1 + C_2|u|^p$ with $1 < p < 2^* - 1$.

Notice that in this case the functional $\mathcal{J}$ is differentiable at every $u \in X$ only along directions $v \in Y := H_0^1(\Omega) \cap L^\infty(\Omega)$ with the derivative given by

$$\langle \mathcal{J}'(u), v\rangle = \int a(x,u)\nabla u \cdot \nabla v + \frac{1}{2}\int a_s'(x,u)|\nabla u|^2 v - \int f(x,u)v.$$

In other words, the Euler–Lagrange problem associated to $\mathcal{J}$ is the following:

$$\begin{aligned} -\text{div}\,(a(x,u)\nabla u) + \tfrac{1}{2}a_s'(x,u)|\nabla u|^2 &= f(x,u), \quad x \in \Omega \\ u &= 0, \qquad\quad x \in \partial\Omega. \end{aligned} \tag{12.14}$$

Hence $\mathcal{J}$ satisfies the assumptions *(a)* and *(b)* of Theorem 12.2.1. In addition, we give sufficient conditions for the condition $(C)$.

**Lemma 12.3.1** *If there exists $s_0 \geq 0$ such that*

$$a'_s(x, s)s \geq 0, \ \text{a.e. } x \in \Omega, \quad \forall s \geq s_0, \tag{12.15}$$

*and there exist $\theta \in \left(0, \frac{1}{2}\right)$ and $\alpha_0 > 0$ such that the inequalities (11.5) and*

$$\left(\frac{p}{2} - 1\right) a(x, s) + \frac{1}{2} a'_s(x, s)s \geq \alpha_0, \quad \text{a.e. } x \in \Omega, \tag{12.16}$$

*hold, then the functional $\mathcal{J}$ satisfies the compactness condition $(C)$.*

*Remark 12.3.2* As in the proof of the $(PS)_c$ condition for the semilinear case studied in Chap. 11 (see Lemma 11.1.1), the outline of the proof consists of two steps. First, we prove that $\{u_n\}$ is bounded and, second, that it possesses a strongly convergent subsequence in $X$. We point out that the second step is more tricky in the quasilinear case (compare it with Lemma 7.1.1).

*Proof* Let $\{u_n\}$ be a sequence in $Y$ satisfying (12.6)–(12.8) for some $\{K_n\} \subset (0, +\infty)$ and $\{\varepsilon_n\} \longrightarrow 0$. We begin by proving that the sequence $\{u_n\}$ is bounded. In order to do this, following the ideas of Lemma 11.1.1, we choose $v = u_n$ as a test function in (12.8) to deduce from (12.7) that

$$|\langle \mathcal{J}'(u_n), u_n \rangle| \leq \varepsilon_n \left[2 + \|u_n\|_X\right].$$

Moreover, by using (12.6), we infer that

$$p\mathcal{J}(u_n) - \langle \mathcal{J}'(u_n), u_n \rangle \leq C + \varepsilon_n \left[2 + \|u_n\|_X\right].$$

Hence, the hypothesis (12.16) means that

$$\begin{aligned}
\alpha_0 \int |\nabla u_n|^2 &\leq \int \left[\left(\frac{1}{2} - \theta\right) a(x, u_n) + \frac{\theta}{2} a'_s(x, u_n)u_n\right] |\nabla u_n|^2 \\
&\leq C + \varepsilon_n \left[2 + \|u_n\|_X\right] + \int \left[F(x, u_n) - \theta u_n f(x, u_n)\right].
\end{aligned}$$

Now, the boundedness of $\{u_n\}$ follows from (11.5) as in the proof for the problem (11.4).

In particular, passing to a subsequence if necessary, we can assume that $\{u_n\}$ is weakly converging to some $u$ in $H_0^1(\Omega)$, and, by Theorem A.4.9, strongly converging in $L^2(\Omega)$ and dominated by a function $\overline{h} \in L^2(\Omega)$, i.e., $|u_n| \leq \overline{h}$ almost everywhere in $\Omega$.

We are going to prove that the sequence $u_n$ is strongly convergent in $H_0^1(\Omega)$ to $u$. Let us introduce the truncature function $T_k$ and $G_k$ given by $T_k(s) = \max\{\min\{s, k\}, -k\}$ and $G_k(s) = s - T_k(s)$, for every $s \in \mathbb{R}$, and then we proceed via the following steps.

*Step* 1. For every fixed $k > s_0$, the sequence $T_k(u_n)$ converges to $T_k(u)$ in $H_0^1(\Omega)$.
*Step* 2. For each $\delta > 0$ , there exist $k_0 \geq s_0$ and $n_0 \in \mathbb{N}$ such that $\|G_k(u_n)\| < \delta$ for every $k \geq k_0$ and $n \geq n_0$.

Indeed, using that $u_n = T_k(u_n) + G_k(u_n)$, we have

$$\begin{aligned}\|u_n - u\| &\le \|u_n - T_k(u)\| + \|T_k(u) - u\| \\ &\le \|T_k(u_n) - T_k(u)\| + \|G_k(u_n)\| + \|T_k(u) - u\|\end{aligned}$$

and Steps 1 and 2 show that the last three terms are arbitrarily small provided that $k$ is sufficiently large, i.e., $u_n$ converges in $H_0^1(\Omega)$ to $u$.

To prove the first step, we follow [32]. We fix $k > s_0$ and take $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ such that $\varphi(0) = 0$. We denote $w_n^k := T_k(u_n) - T_k(u)$. It is easy to verify that the sequence $\{\varphi(w_n^k)\}$ is

- weakly converging to zero in $H_0^1(\Omega)$,
- converging to zero almost everywhere in $\Omega$ and
- strongly converging to zero in $L^q(\Omega)$ for every $q \in [1, +\infty)$.

Using (12.1) and (12.15), we deduce that if $k > s_0$, then

$$\begin{aligned}\alpha \int |\nabla w_n^k|^2 \varphi'(w_n^k) \le& \int a(x, u_n)|\nabla w_n^k|^2 \varphi'(w_n^k) \\ &+ \int_{k<u_n} \frac{a_s'(x, u_n)}{2} |\nabla u_n|^2 \varphi(k - T_k(u)) \\ &+ \int_{u_n<-k} \frac{a_s'(x, u_n)}{2} |\nabla u_n|^2 \varphi( - k - T_k(u))\end{aligned}$$

and thus

$$\begin{aligned}\alpha \int |\nabla w_n^k|^2 \varphi'(w_n^k) \le& \int a(x, u_n)\nabla u_n \cdot \nabla(w_n^k)\varphi'(w_n^k) \\ &- \int_{|u_n|>k} a(x, u_n)\nabla u_n \cdot \nabla(w_n^k)\varphi'(w_n^k) \\ &- \int a(x, u_n)\nabla T_k(u) \cdot \nabla(w_n^k)\varphi'(w_n^k) \\ &+ \int \frac{a_s'(x, u_n)}{2} |\nabla u_n|^2 \varphi(w_n^k) \\ &- \int_{|u_n|\le k} \frac{a_s'(x, u_n)}{2} |\nabla u_n|^2 \varphi(w_n^k).\end{aligned}$$

Now, taking $v_n = \varphi(w_n^k)$ as a test function in (12.8), we obtain

$$\left| \int a(x, u_n)\nabla u_n \cdot \nabla(\varphi(w_n^k)) + \frac{a_s'(x, u_n)}{2} |\nabla u_n|^2 \varphi(w_n^k) - f(x, u_n)\varphi(w_n^k) \right| \le \overline{\varepsilon}_n,$$

where $\overline{\varepsilon}_n > 0$, $\overline{\varepsilon}_n \to 0$ and hence, by (12.13), we deduce that

$$\begin{aligned}\alpha \int |\nabla w_n^k|^2 \varphi'(w_n^k) \leq \overline{\varepsilon}_n &+ \left| \int_{|u_n|>k} a(x,u_n)\nabla u_n \cdot \nabla(w_n^k)\varphi'(w_n^k) \right| \\ &+ \left| \int a(x,u_n)\nabla T_k(u) \cdot \nabla w_n^k \varphi'(w_n^k) \right| \\ &+ \gamma \int |\nabla w_n^k|^2 \varphi(w_n^k) + \gamma \int |\nabla T_k(u)|^2 \varphi(w_n^k) \\ &+ \left| \int f(x,u_n)\varphi(w_n^k) \right|.\end{aligned}$$

By Corollary A.1.13 and the definition of $w_n^k$,

$$\int_{|u_n|>k} a(x,u_n)\nabla u_n \cdot \nabla w_n^k \varphi'(w_n^k) = -\int_{|u_n|>k} a(x,u_n)\nabla u_n \cdot \nabla T_k(u)\varphi'(w_n^k).$$

The weak convergence of $\nabla u_n$ to $u$ and the strong one of $a(x,u_n)\nabla T_k(u)\varphi'(w_n^k)$ to zero in $L^2(\Omega)$ imply then that

$$\lim_{n\to\infty} \int_{|u_n|>k} a(x,u_n)\nabla u_n \cdot \nabla w_n^k \varphi'(w_n^k) = 0.$$

Similarly,

$$\lim_{n\to\infty} \int a(x,u_n)\nabla T_k(u) \cdot \nabla w_n^k \varphi'(w_n^k) = 0$$

and

$$\lim_{n\to\infty} \int |\nabla T_k(u)|^2 \varphi(w_n^k) = \lim_{n\to\infty} \int f(x,u_n)\varphi(w_n^k) = 0,$$

and consequently, we have

$$\lim_{n\to\infty} \int |\nabla w_n^k|^2 \left[\alpha\varphi'(w_n^k) - \gamma\varphi(w_n^k)\right] = 0.$$

Choosing $\varphi(s) = se^{\eta s^2}$ with $\eta > 0$ large enough, it is easy to verify that $\alpha\varphi'(s) - \gamma\varphi(s) \geq \frac{\alpha}{2}$ for every $s \in \mathbb{R}$ and, therefore, we deduce that

$$\lim_{n\to\infty} \int |\nabla w_n^k|^2 = 0,$$

i.e., we prove the strong convergence in $H_0^1(\Omega)$ of $T_k(u_n)$ to $T_k(u)$, and Step 1 is concluded.

With respect to the proof of Step 2, observe that taking $v = G_k(u_n)$ as a test function in (12.8) and using (12.7), we have that

$$\delta_n := \int a(x,u_n)|\nabla G_k(u_n)|^2 + \frac{1}{2}\int a'(x,u_n)G_k(u_n)|\nabla G_k(u_n)|^2 - \int f(x,u_n)G_k(u_n)$$

is a sequence of numbers which converges to zero. By using (12.1), (12.15) and $k \geq s_0$ we get

$$\alpha \int |\nabla G_k(u_n)|^2 \leq \int a(x,u_n)|\nabla G_k(u_n)|^2 + \frac{1}{2}\int a'(x,u_n)G_k(u_n)|\nabla G_k(u_n)|^2$$
$$\leq \delta_n + \int f(x,u_n)G_k(u_n).$$

Moreover, by the subcritical growth condition $(ii)$ and the Sobolev embedding theorem (see Corollary A.4.10) we also get

$$\begin{aligned}\int f(x,u_n)G_k(u_n) &\leq C_2\int |G_k(u_n)| + C_2\int |u_n|^p\,|G_k(u_n)| \\ &\leq C_1|\Omega_{n,k}|^{\frac{1}{2}}\|G_k(u_n)\|_2 \\ &\quad + C_2\|G_k(u_n)\|_{2^*}\left(\int_{\Omega_{n,k}} |u_n|^{\overline{2}p}\right)^{\frac{1}{2}} \\ &\leq C_3\|G_k(u_n)\|\,|\Omega_{n,k}|^{\frac{1}{2}} \\ &\quad + C_4\|G_k(u_n)\|\,\|u_n\|^{p/2^*}\left(|\Omega_{n,k}|\right)^{\left(\frac{1}{2}-\frac{p}{2^*}\right)}\end{aligned}$$

where $\overline{2} = 2N/(N+2)$ is the Hölder conjugate exponent of $2^*$ (see Notation). By the boundedness of $u_n$ in $H_0^1(\Omega)$ and by the Young inequality, we obtain

$$\begin{aligned}\int f(x,u_n)G_k(u_n) &\leq C_5\left[|\Omega_{n,k}|^{\frac{1}{2}} + |\Omega_{n,k}|^{\left(1-\frac{p}{2^*}\right)\frac{1}{2}}\right]\|G_k(u_n)\| \\ &\leq C_5\left[|\Omega_{n,k}|^{\frac{1}{2}} + |\Omega_{n,k}|^{\left(1-\frac{p}{2^*}\right)\frac{1}{2}}\right]\left(\frac{\|G_k(u_n)\|^2}{2} + \frac{1}{2}\right).\end{aligned}$$

Since $\Omega_{n,k} \subset \{x \in \Omega : \overline{h}(x) > k\}$, then $\lim_{k\to\infty} |\Omega_{n,k}| = 0$, uniformly in $n \in \mathbb{N}$. Therefore, for each $\delta > 0$, there exists $k_0 \geq s_0$ such that for every $k \geq k_0$ and $n \in \mathbb{N}$,

$$\frac{C_5}{2}\left[|\Omega_{n,k}|^{\frac{1}{2}} + |\Omega_{n,k}|^{\left(1-\frac{p}{2^*}\right)\frac{1}{2}}\right] < \frac{\alpha}{2}.$$

Consequently,

$$\frac{\alpha}{2}\|G_k(u_n)\|^2 \leq \delta_n + \frac{C_5}{2}\left[|\Omega_{n,k}|^{\frac{1}{2}} + |\Omega_{n,k}|^{\left(1-\frac{p}{2^*}\right)\frac{1}{2}}\right],$$

and there exists $n_0 \in \mathbb{N}$ such that

$$\|G_k(u_n)\| < \delta, \quad \forall k \geq k_0, \quad \forall n \geq n_0.$$

This concludes Step 2 and the proof of the lemma. □

**Theorem 12.3.3** *Assume that the preceding conditions* $(i)$ *and (ii) are satisfied. If, in addition, there exist* $s_0 > 0$, $\theta \in \left(0, \frac{1}{2}\right)$ *and* $\alpha_0 > 0$ *such that the inequalities* (11.5), (12.15) *and* (12.16) *hold, then problem* (12.14) *has at least one nontrivial solution.*

*Proof* We begin by proving that if $E = X = H_0^1(\Omega)$, then the functional $\mathcal{J}$ satisfies the conditions $(J1)$ and $(J2)$ introduced in Sect. 5.3. The verification is divided into two steps:

*Step* 1. First we show that $u = 0$ is a strict local minimum of $\mathcal{J}$. Indeed, by hypotheses (i) and (ii), we deduce that for fixed $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(x, s) \leq \alpha_1 \varepsilon s^2 + C_\varepsilon s^{p+1}, \quad \forall s \geq 0.$$

Then $\mathcal{F}(u) := \int F(x, u) = o\left(\|u\|^2\right)$ at $u = 0$ and, by (12.1), we obtain

$$\mathcal{J}(u) \geq \alpha \|u\|^2 - o\left(\|u\|^2\right)$$

from which one easily deduce the existence of positive constants $\rho$, $\overline{R}$ such that

$$\mathcal{J}(u) \geq \rho > 0 \qquad \text{for } \|u\| = \overline{R} > 0 \tag{12.17}$$

and $(J1)$ is verified.

*Step* 2. To verify $(J2)$, observe that, by (12.1) and (11.5) (which implies (11.6)), we have for $t > 0$

$$\mathcal{J}(t\varphi_1) \leq \frac{\beta}{2} t^2 \|\varphi_1\|^2 - C_4 t^{1/\theta} \|\varphi_1\|^{1/\theta} + C_5.$$

Thus, there exists $t_0 > \frac{\overline{R}}{\|\varphi_1\|}$ such that $\mathcal{J}(t_0\varphi_1) < 0$ and condition $(J2)$ is satisfied.

Now, in order to apply Theorem 12.2.1, take $X = H_0^1(\Omega)$ and $Y = H_0^1(\Omega) \cap L^\infty(\Omega)$ endowed with the norm $\|\cdot\|_Y = \|\cdot\|_\infty + \|\cdot\|$, and $e = t_0\varphi_1$. Moreover, let

$$\Gamma = \{\gamma : [0, 1] \longrightarrow (Y, \|\cdot\|_Y) \ : \ \gamma \text{ is continuous and } \gamma(0) = 0, \quad \gamma(1) = e\}.$$

Observe that every $\gamma \in \Gamma$ is continuous from $[0, 1]$ to $H_0^1(\Omega)$, so that, since $\|t_0\varphi_1\| \geq 2\overline{R}$, there exists $\bar{t} \in [0, 1]$ such that $\|\gamma(\bar{t})\| = \overline{R}$. Thus, by (12.17),

$$c \equiv \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{J}(\gamma(t)) \geq \rho > \max\{\mathcal{J}(0), \mathcal{J}(t_0\varphi_1)\} = 0,$$

and hypothesis (12.5) holds.

In addition, Lemma 12.3.1 implies that $(C)$ is verified, and Theorem 12.2.1 implies that there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $\mathcal{J}(u) = c > 0$ and $\langle \mathcal{J}'(u), v \rangle = 0$ for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. $\square$

## 12.4 Some Nonvariational Quasilinear Problems

We will see now how the topological methods developed in Chaps. 4 and 6 can also be applied to study quasilinear problems. Specifically, let $\Omega$ be an open and bounded subset of $\mathbb{R}^N$ $(N \geq 3)$, $\lambda \geq 0$, $1 < p < \frac{N+2}{N-2}$ and $0 \leq h \in L^{\frac{2N}{N+2}}(\Omega)$. Consider the (nonvariational) boundary value problem

$$\begin{aligned} -\Delta u + g(u)|\nabla u|^2 &= \lambda |u|^{p-1}u + h(x), && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned} \tag{12.18$_\lambda$}$$

for a suitable non-negative continuous function $g : [0, +\infty) \longrightarrow [0, +\infty)$. We look for positive solutions of (12.18$_\lambda$), i.e., $u \in H_0^1(\Omega)$ such that $u > 0$ a.e. $x \in \Omega$, $g(u)|\nabla u|^2 \in L^1(\Omega)$ and

$$\int \nabla u \cdot \nabla \varphi + \int g(u)|\nabla u|^2 \varphi = \lambda \int u^p \varphi + \int h\varphi, \tag{12.19}$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

The following lemma concerning the regularity of the solutions will be useful in the sequel. It can be easily deduced by using the Stampacchia technique [82], as in the proof of Lemma 11.2.3. The details are left to the reader (see Exercise 50).

**Lemma 12.4.1** *Assume that $h \in L^q(\Omega)$ with $q > \frac{N}{2}$. If $u \in H_0^1(\Omega)$ is a solution for* (12.18$_\lambda$)*, then $u$ belongs to $L^\infty(\Omega)$.*

Now we give sufficient conditions to ensure that problem (12.18$_\lambda$) satisfies the uniform strong maximum principle in compactly embedded domains; that is, for every $\omega \subset\subset \Omega$ there exists a positive constant (independent from $\lambda$) which is a lower bound in $\omega$ of any solution of (12.18$_\lambda$).

**Proposition 12.4.2** *Suppose that $0 \lneqq h \in L^q(\Omega)$, $q > N/2$. Then for every $\omega \subset\subset \Omega$ there exists $L_\omega > 0$ such that*

$$u(x) \geq L_\omega, \quad a.e. \quad x \in \omega,$$

*for every super-solution $u \in H_0^1(\Omega)$ of* (12.18$_\lambda$) *(with $\lambda$ any positive constant).*

*Proof* For every $s \in \mathbb{R}$, $T_1(s) = \max\{\min\{s, 1\}, -1\}$. Taking into account that $\lambda s^p + h(x) \geq T_1(h(x))$, for all $s \geq 0$, every solution $u \in H_0^1(\Omega)$ is a super-solution for the problem

$$\begin{aligned} -\Delta v + g(v)|\nabla v|^2 &= T_1(h(x)), && x \in \Omega, \\ v &= 0, && x \in \partial\Omega. \end{aligned}$$

By Theorems 12.1.1 and 12.1.2, this problem has a unique continuous solution $v \in H_0^1(\Omega) \cap C(\overline{\Omega})$. Using that $v \in C(\overline{\Omega})$ and $v > 0$ in $\Omega$, if $\omega \subset\subset \Omega$ we infer the existence of $L_\omega > 0$ such that $v(x) \geq \min_\omega v = L_\omega$. By the comparison principle given in Theorem 12.1.2, we deduce that $u \geq v \geq L_\omega$, and the proof is concluded. ◻

Observe that with the notation of Sect. 12.1, ($12.18_\lambda$) can be rewritten[1] as a fixed point problem, namely,

$$u = K^g_\lambda(u),$$

with $K^g_\lambda(u) = K^g(\lambda, u)$. The compactness of $K^g(\lambda, w)$ (see Proposition 12.1.3) allows us to apply the Leray–Schauder degree techniques to study the existence of "continua of solutions" of ($12.18_\lambda$).

Consider the solution set, i.e.,

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times H^1_0(\Omega) \,:\, u = K^g_\lambda(u)\}.$$

We state the existence of solutions by proving the existence of a global continuum in $\Sigma$ which emanates from the unique solution of ($12.18_0$) and by establishing its asymptotic behavior. We point out that in the semilinear case ($g \equiv 0$), given $h \gneqq 0$ and $p > 1$, there exists $\lambda^* > 0$ such that ($12.18_\lambda$) has no positive solution for every $\lambda > \lambda^*$, i.e., $\mathrm{Proj}_{[0,+\infty)}\Sigma$ is bounded. On the contrary, the quasilinear case is quite different. Indeed, we give sufficient conditions (see (12.20) below) to ensure that $\mathrm{Proj}_{[0,+\infty)}\Sigma$ is unbounded. The role of these conditions is to provide for every compact set $\Lambda$ of $\lambda$'s the existence of suitable a priori bounds of the $H^1_0(\Omega)$-norm of solutions of ($12.18_\lambda$) with $\lambda \in \Lambda$, i.e., to establish that the $\lambda$-slice $\Sigma_\lambda = \{u \in H^1_0(\Omega) \,:\, (\lambda, u) \in \Sigma\}$ is bounded.

**Theorem 12.4.3** *Consider $p \in (1, 2^* - 1)$, $0 \lneqq h \in L^q(\Omega)$, $q > N/2$ and assume that $g \geq 0$ is continuous in the interval $[0, +\infty)$.*

(i) *If $1 < p < 2$ and, for some constants $s_1, c > 0$ and $0 \leq \gamma < 2 - p$, $g$ satisfies*

$$g(s) \geq \frac{c}{s^\gamma}, \quad \forall s \geq s_1, \tag{12.20}$$

*then problem ($12.18_\lambda$) admits a positive solution for every $\lambda \in [0, +\infty)$.*

(ii) *If there are $s_0, \delta_0 > 0$ such that*

$$\frac{s^p}{\int_0^s e^{\int_r^s g(t)dt} dr} \geq \delta_0, \quad \forall s > s_0, \tag{12.21}$$

*then there exist $\lambda^*, \lambda_* > 0$ such that ($12.18_\lambda$) admits a positive solution for every $\lambda \in [0, \lambda_*)$ and admits no positive solution for $\lambda > \lambda^*$.*

*Proof* First, we prove that there exists an unbounded continuum $S \subset \Sigma$ which contains $(0, u_0)$, where $u_0$ is the unique solution of ($12.18_0$). In order to do this, we compute the index of the solution $u_0 \in H^1_0(\Omega)$ for ($12.18_0$) by showing that $i(K^g_0, u_0) = 1$. Indeed, by Theorems 12.1.1 and 12.1.2 let $U(t)$ be the unique solution of

$$\begin{aligned} -\Delta u + t g(u)|\nabla u|^2 &= h(x), \quad && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned} \tag{12.22}$$

[1] Compare this approach with the one in the work by Ruiz and Suárez [79], for $g \equiv 1$ and a logistic nonlinearity, where the authors combine regularity in $C^1(\overline{\Omega})$ with the properties of the inverse $K$ of the Laplacian operator in $C(\overline{\Omega})$ in order to use bifurcation techniques.

and define $H : [0,1] \times H_0^1(\Omega) \to H_0^1(\Omega)$ by setting $H(t,w) = U(t)$ for every $(t,w) \in [0,1] \times H_0^1(\Omega)$. Observe that $H(1,w) = U(1) = K_0^g(w) = u_0$, while $H(0,w) = U(0) = K(h(x))$ and it is well known that $i(K(h(x)), U(0)) = 1$. By Proposition 12.1.3 we deduce that $H$ is compact. In addition, using that $g \geq 0$, and taking $u$ as a test function in (12.22), we have $\int |\nabla u|^2 \leq \int h\, u$. The Hölder and Sobolev inequalities with $\mathcal{S} = \sup\{\|u\|_{2^*} : \|u\| = 1\}$ the Sobolev constant (see Notation) imply $\|U(t)\| \leq \mathcal{S}\|h\|_{2N/(N+2)}$, for every $t \in [0,1]$. Hence, choosing $R > \mathcal{S}\|h\|_{2N/(N+2)}$, we obtain $u \neq H(t,u)$ for every $t \in [0,1]$ and $u \in H_0^1(\Omega)$ with $\|u\| \geq R$, and we can apply the homotopy invariance of the degree to conclude that

$$\begin{aligned} i(K_0^g, u_0) &= i(H(1,\cdot), U(1)) = i(H(0,\cdot), U(0)) \\ &= i(K(h(x)), U(0)) = 1, \end{aligned}$$

and the claim has been proved. The existence of an unbounded continuum $S \subset \Sigma$ follows now from Theorem 4.4.1.

The unboundedness of the continuum $S$ implies that one of the projections of $S$, either its projection $\text{Proj}_{[0,+\infty)} S$ on the $\lambda$-axis or its projection $\text{Proj}_{H_0^1(\Omega)} S$ on the $H_0^1(\Omega)$-axis, is an unbounded set. We will see that in case i) the former projection is unbounded. More precisely, for every compact set $\Lambda$ of $\lambda$'s, we will show the existence of suitable a priori bounds of the $H_0^1(\Omega)$-norm of solutions of $(12.18_\lambda)$ with $\lambda \in \Lambda$. This will imply that the $\lambda$-slice $S_\lambda = \{u \in H_0^1(\Omega) : (\lambda, u) \in S\}$ is bounded. On the other hand, in case ii) it is the later projection which is unbounded since, as we will see below, there exists $\lambda^* > 0$ such that $(12.18_\lambda)$ has no positive solutions for $\lambda > \lambda^*$. Clearly, this will conclude the proof of the theorem.

(i) Since $1 < p < 2$ and (12.20) holds with $\gamma < 2 - p$, we may construct a continuous and non-negative function $g_0(s)$ such that $g_0(s) = 0$ for every $s < \frac{s_0}{2}$, $g_0(s) = \frac{c}{s^\gamma}$ for every $s > s_0$ and $g(s) \geq g_0(s)$ for every $s > 0$. We also define the function $\varphi(s)$ given by

$$\varphi(s) = \int_0^s \exp\left(-\int_v^s g_0(t)dt\right)dv, \quad \forall s \geq 0.$$

It is elementary to prove that

1. $0 \leq \varphi(s) \leq s$ for every $s \in (0, +\infty)$.
2. $\varphi'(s) + g_0(s)\varphi(s) = 1$ for every $s \in (0, +\infty)$.
3. There exists $\sigma > 0$ with $g_0(s)\varphi(s) \leq \sigma$ for every $s > 0$.

Let $u \in H_0^1(\Omega)$ be a positive solution of $(12.18_\lambda)$ for some $\lambda > 0$. Observe that, using 3., $g_0(u)\varphi(u) \in L^\infty(\Omega)$, which, by 2., implies that $\nabla\varphi(u) \in L^2(\Omega)$. Taking into account that, by 1., we have $\varphi(u) \leq u$ we deduce that $\varphi(u) \in H_0^1(\Omega)$. Hence we can take $\varphi(u)$ as a test function and using that $g_0(s) \leq g(s)$ we obtain

$$\begin{aligned} \|u\|^2 &\overset{(2)}{=} \int |\nabla u|^2 \left(\varphi'(u) + g_0(u)\varphi(u)\right) \\ &\leq \int \left(\nabla u \cdot \nabla\varphi(u) + g(u)|\nabla u|^2\varphi(u)\right) \\ &= \int (\lambda u^p + h)\varphi(u). \end{aligned}$$

Since $\varphi(s)s^{-\gamma}$ is bounded from above near infinity (by 3 and by the construction of $g_0$) and near zero (by the definition of $\varphi$ and $\gamma < 2 - p < 1$), there exists $C > 0$ such that $\varphi(s) \leq Cs^{\gamma}$ for every $s > 0$. Therefore, dividing the previous inequality by $\|u\|^{p+\gamma}$, setting $z = u/\|u\|$ and using the Hölder and Sobolev inequalities we get

$$\begin{aligned}\|u\|^{2-p-\gamma} &\leq C\lambda \int z^{p+\gamma} + \int \frac{hz}{\|u\|^{p+\gamma-1}} \\ &\leq C_1\lambda + \frac{C_2}{\|u\|^{p+\gamma-1}}\|h\|_{2N/(N+2)},\end{aligned}$$

for some $C_1, C_2 > 0$. Consequently, we deduce that if $\lambda$ is bounded, then the norm $\|u\|$ is also bounded. Therefore the proof of case i) is done.

(ii) For $\omega \subset\subset \Omega$ we denote by $\chi_\omega(x)$ the characteristic function of $\omega$ and consider the first eigenvalue (resp. eigenfunction) $\mu_\omega$ (resp, $\phi_\omega$) associated to the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \chi_\omega(x) u, & x \in \Omega, \\ \quad u = 0, & x \in \partial\Omega. \end{cases}$$

We show that a necessary condition for the existence of solution $u \in H_0^1(\Omega)$ of $(12.18_\lambda)$ is $\mu_\omega \geq \lambda c$, for a suitable positive constant $c$. To do that, consider a sequence of functions $0 \leq \phi_n \in C_c^\infty(\Omega)$ converging in $H_0^1(\Omega)$ to $\phi_\omega$. Taking $\varphi(u) = e^{-\int_1^{T_k(u)} g(t)dt}\phi_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as a test function in $(12.18_\lambda)$ and using the fact that $h \geq 0$, we get

$$\begin{aligned}&\int \nabla u \cdot \nabla \phi_n e^{-\int_1^{T_k(u)} g(t)dt} + \int_{\{u \geq k\}} g(u)|\nabla u|^2 e^{-\int_1^k g(t)dt}\phi_n \\ &\geq \lambda \int u^p e^{-\int_1^{T_k(u)} g(t)dt}\phi_n.\end{aligned}$$

Taking limits, firstly as $k$ tends to $\infty$ (using the Fatou lemma) and secondly as $n$ goes to $\infty$ (using the $H_0^1(\Omega)$-convergence of $\phi_n$ to $\phi_\omega$ and the Lebesgue theorem), we have

$$\int \nabla u \cdot \nabla \phi_\omega e^{-\int_1^u g(t)dt} \geq \lambda \int u^p e^{-\int_1^u g(t)dt}\phi_\omega.$$

On the other hand, taking

$$\psi(u) = \frac{u^p e^{-\int_1^u g(t)dt}}{\int_0^u e^{-\int_1^s g(t)dt} ds}$$

for $u > 0$ and choosing $w = \displaystyle\int_0^u e^{-\int_1^s g(t)dt} ds \in H_0^1(\Omega)$ as a test function in the equation satisfied by $\phi_\omega$, we find

$$\begin{aligned}\mu_\omega \int \chi_\omega(x) w \phi_\omega = \int \nabla w \cdot \nabla \phi_\omega &\geq \lambda \int w \phi_\omega \psi(u) \\ &\geq \lambda \int \chi_\omega(x) w \phi_\omega \psi(u).\end{aligned}$$

Using Proposition 12.4.2, there exists $L_\omega > 0$ such that $u(x) > L_\omega$ a.e. $x \in \omega$. Moreover, by condition (12.21), $c := \inf_{s\in[L_\omega,\infty)} \psi(s) > 0$ and we conclude that

$$\mu_\omega \int w\phi_\omega \geq \lambda c \int w\phi_\omega,$$

that is, $\mu_\omega \geq \lambda c$, as desired. Choosing $\lambda^* = \mu_\omega / c$ we conclude the proof of case (ii). □

*Remark 12.4.4* Moreover, if in addition to the hypotheses (i) of the above theorem, $g$ satisfies that there exists $q \leq p$ such that

$$g(s) \leq C\left(s^q + \frac{1}{s}\right), \quad \forall s > 0, \tag{12.23}$$

then $\|u_n\| \to +\infty$ for every sequence $(\lambda_n, u_n)$ in $\Sigma$ with $\lambda_n \to +\infty$. Indeed, if $(\lambda_n, u_n) \in \Sigma$ and for $0 \leq \varphi \in C_0^\infty(\Omega)$ we take $\dfrac{\varphi}{u_n^q}$ as a test function (it is an admissible test function due to Proposition 12.4.2), we have

$$\int \nabla u_n \cdot \frac{\nabla\varphi}{u_n^q} - q\int \frac{|\nabla u_n|^2}{u_n^{q+1}}\varphi + \int \frac{g(u_n)}{u_n^q}|\nabla u_n|^2\varphi - \int h\frac{\varphi}{u_n^q} = \lambda_n \int u_n^{p-q}\varphi,$$

and thus

$$\int \nabla u_n \cdot \frac{\nabla\varphi}{u_n^q} + \int \frac{g(u_n)}{u_n^q}|\nabla u_n|^2\varphi \geq \lambda_n \int u_n^{p-q}\varphi.$$

By Proposition 12.4.2, $u_n$ is uniformly away from zero in supp $\varphi$ and, therefore, using (12.23) we deduce that $\dfrac{g(u_n)}{u_n^q}$ is bounded from above in supp $\varphi$ and the sequence $\int u_n^{p-q}\varphi$ is also away from zero. Therefore, if $u_n$ is bounded in $H_0^1(\Omega)$, the left-hand side of the above equality is bounded from above, and thus $\lambda_n$ has to also be bounded. In this way, the remark follows.

The last part of the section is devoted to studying the case $h \equiv 0$.

**Theorem 12.4.5** *Assume $h \equiv 0$ and suppose that $g \geq 0$ is continuous in the interval* $[0, +\infty)$. *If $p > 1$ and there is a continuous non-positive function $\overline{g} \in L^1(0, +\infty)$ such that*

$$g(s) \geq \overline{g}(s) + \frac{p}{s}, \quad \forall s \geq 1, \tag{12.24}$$

*then there exists $\lambda^* > 0$ such that* (12.18$_\lambda$) *has no solution for $\lambda < \lambda^*$.*

*Remark 12.4.6* It is shown by Orsina and Puel [73] (for the case $h \equiv 0$) that a suitable change of variables reduces the quasilinear equation to a semilinear one. In this way the authors prove that if $g \in L^1(0, +\infty)$, then there exists a positive solution for every $\lambda > 0$, while if $g(t)t \geq q > p$ for $t \gg 1$, then there exists a positive solution for $\lambda > 0$ large enough and no positive solution if $\lambda > 0$ is sufficiently small. The above improvement as well as Theorem 12.4.3 are contained in [22] and show that the topological methods help us to understand the true role of the different hypotheses imposed on the behavior of the nonlinearity $g$, revealing the different effects that take place in the solution set $S$.

*Proof* We consider the function

$$\varphi(s) = \int_0^{T_1(s)} \exp\left(\int_s^t g(r)dr\right)dt, \quad \forall s > 0.$$

which satisfies the following.

1. $\varphi'(s) + g(s)\varphi(s) = [T_1'(s)]^2$ for every $0 < s \neq 1$.
2. There exists a positive constant $C$ such that $s^p\varphi(s) \leq C[T_1(s)]^2$ for all $s > 0$. Indeed, this is trivial for $s \leq 1$, while for $s > 1$, taking into account that $\overline{g} \leq 0$ and (12.24), we have

$$\begin{aligned} s^p\varphi(s) &= \int_0^1 s^p \exp\left(\int_s^1 g(r)dr + \int_1^t g(r)dr\right)dt \\ &= \varphi(1)\exp\left(\int_s^1 \left(g(r) - \frac{p}{r}\right)dr\right) \\ (12.24) &\leq \varphi(1)\exp\left(-\int_1^{+\infty} \overline{g}(r)dr\right) \equiv C \leq C[T_1(s)]^2. \end{aligned}$$

If $u \in H_0^1(\Omega)$ is a positive solution of $(12.18_\lambda)$ and we take $\varphi(u) \in H_0^1(\Omega)$ as a test function we deduce from the above items that

$$\begin{aligned} \mu_1 \int [T_1(u)]^2 &\leq \int |\nabla T_1(u)|^2 = \int |\nabla u|^2(\varphi'(u) + g(u)\varphi(u)) \\ &\leq \lambda \int u^p\varphi(u) \leq C\lambda \int [T_1(u)]^2, \end{aligned}$$

and the proof is concluded by taking $\lambda^* = \frac{\mu_1}{C}$. □

# Chapter 13
# Stationary States of Evolution Equations

This final chapter deals with the existence of ground and bound states of nonlinear Schrödinger (NLS) equations. Semiclassical states are discussed in Sect. 13.2. Systems of coupled NLS equations are handled in Sects. 13.3 and 13.4.

When dealing with elliptic equations on unbounded domains the main problem is the $(PS)$ condition. We show how one can bypass this difficulty in a few specific cases.

However, the study of problems on unbounded domains is out of the scope of this book. The interested reader is referred, e.g., to [15], which contains several references.

## 13.1 Soliton States to Stationary NLS Equations

Elliptic equations on all $\mathbb{R}^N$ arise as stationary states of evolution equations such as the time-independent solution of the nonlinear wave equation

$$u_{tt} - \Delta u + \lambda u = u^p.$$

As a second example, we can consider the NLS equation

$$-i\,\psi_t = \Delta\psi + a\psi + |\psi|^{p-1}\psi, \tag{13.1}$$

where $i$ denotes the imaginary unit and $\psi = \psi(t,x)$ is complex valued. In (13.1), the *ansatz* $\psi(t,x) = e^{-i\omega t}u(x)$, with $u(x) \in \mathbb{R}$, yields for $u$ the equation

$$-\Delta u + \lambda u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{13.2}$$

where $\lambda = a - \omega$. We will assume that $\lambda > 0$ and $1 < p < 2^* - 1$, where $2^*$ is given by (7.3). We look for solutions $u > 0$ of (13.2) such that $u \in E := H^1(\mathbb{R}^N)$. These solutions verify

$$\int u^2 < \infty, \qquad \int |\nabla u|^2 < \infty,$$

and are called *bound states* of (13.2). Among the bound states, solutions with minimal energy have a particular interest. They are called *ground states*.

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82, DOI 10.1007/978-0-8176-8114-2_13, 

Let $u \in E$ be a solution of

$$-\Delta u + u = |u|^{p-1}u. \tag{13.3}$$

A straight calculation shows that

$$u_\lambda(x) = \lambda^{1/(p-1)}u(\lambda^{1/(p-1)}x) \tag{13.4}$$

solves (13.2). Therefore in the sequel we will look for solutions of (13.3), which can be found as critical points of

$$\mathcal{J}(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int |u|^{p+1}, \quad u \in E,$$

where $\|u\|^2 = \int \left[|\nabla u|^2 + u^2\right]$. With this notation, a ground state of (13.3) is a solution $z > 0$ such that

$$\mathcal{J}(z) = \min\{\mathcal{J}(u) : u \in E,\ \mathcal{J}'(u) = 0\}.$$

There are several methods that can be used to find a ground state. We will employ the *Nehari natural constraint*. Let us introduce the functional

$$\mathcal{G}(u) = (\mathcal{J}'(u) \mid u) = \|u\|^2 - \int |u|^{p+1},$$

and consider the set

$$\mathcal{N} = \{u \in E \setminus \{0\} : \mathcal{G}(u) = 0\}.$$

Since, obviously, any positive solution of (13.3) belongs to $\mathcal{N}$, then any $z$ such that

$$\mathcal{J}(z) = \min\{\mathcal{J}(u) : u \in \mathcal{N}\}$$

is a ground state (see also Lemma 13.1.1). Some of the main features of $\mathcal{N}$ are collected below.

$(N.1)$ $\exists\, r > 0$ such that $\|u\| \ge r$ for all $u \in \mathcal{N}$.

*Proof* From $\mathcal{G}''(0)[v, v] = 2\|v\|^2$ it follows that $\exists\, r > 0$ such that $\mathcal{G}(u) > 0$ for all $u$ with $0 < \|v\| \le r$, yielding $(N.1)$. □

$(N.2)$ $\inf_{\mathcal{N}} \mathcal{J}(u) \ge (\frac{1}{2} - \frac{1}{p+1})r^2$.

*Proof* For all $u \in \mathcal{N}$ there holds

$$\mathcal{J}(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int |u|^{p+1} = (\tfrac{1}{2} - \tfrac{1}{p+1})\|u\|^2, \tag{13.5}$$

and $(N.1)$ implies $\mathcal{J}(u) \ge (\frac{1}{2} - \frac{1}{p+1})r^2$. □

$(N.3)$ For all $u \in \mathcal{N}$ it holds that $(\mathcal{G}'(u) \mid u) \le (1-p)r^2 < 0$.

*Proof* One has that

$$(\mathcal{G}'(u) \mid u) = 2\|u\|^2 - (p+1)\int |u|^{p+1}.$$

Since on $\mathcal{N}$, $u \neq 0$ and $\int |u|^{p+1} = \|u\|^2$, we infer that $(\mathcal{G}'(u) \mid u) = (1-p)\|u\|^2$, and $(N.3)$ follows from $(N.1)$. □

$(N.4)$ $\mathcal{N}$ is a smooth manifold diffeomorphic to the unit sphere $S$ of $E$.

*Proof* From $(N.3)$ it follows, in particular, that $\mathcal{G}'(u) \neq 0$ for all $u \in \mathcal{N}$ and this proves that $\mathcal{G}^{-1}(0) \setminus \{0\}$ is a smooth manifold (of codimension 1) in $E$. This and $(N.1)$ imply that $\mathcal{N}$ is also a smooth manifold. Moreover, for all $v \in E$, $v \neq 0$, $\mathcal{G}(tv) = 0$ if and only if $t^2\|v\|^2 - t^{p+1}\int |v|^{p+1}$. This means that

$$tv \in \mathcal{N} \iff t^{p-1} = \frac{\|v\|^2}{\int |v|^{p+1}}, \tag{13.6}$$

proving that $\mathcal{N} \simeq S$. □

We are now in position to prove the following lemma.

**Lemma 13.1.1** *Any critical point of $\mathcal{J}$ constrained on $\mathcal{N}$ is a critical point of $\mathcal{J}$ on $E$.*

*Proof* Let $z \in \mathcal{N}$ be a critical point of $\mathcal{J}$ constrained on $\mathcal{N}$, i.e., satisfying $\nabla_{\mathcal{N}}\mathcal{J}(z) = 0$. By the Lagrange multiplier rule (see also Remark 5.3.5) there exists $\mu \in \mathbb{R}$ such that $\mathcal{J}'(z) = \mu\mathcal{G}'(z)$. Taking the scalar product with $z$, we find

$$(\mathcal{J}'(z) \mid z) = \mu(\mathcal{G}'(z) \mid z). \tag{13.7}$$

From $(N.3)$ it follows that $(\mathcal{G}'(z) \mid z) \neq 0$. On the other hand $(\mathcal{J}'(z) \mid z) = \mathcal{G}(z) = 0$ and therefore (13.7) implies that $\mu = 0$, whence $\mathcal{J}'(z) = \mu\mathcal{G}'(z) = 0$. □

After these preliminaries, we can prove the existence of a ground state of (13.3).

**Theorem 13.1.2** *If $1 < p < 2^* - 1$, then (13.3) has a positive ground state $U$, which is radially symmetric.*

*Proof* If $N = 1$ an elementary phase plane analysis shows that (13.3) has a unique radially symmetric, radially decreasing ground state $U$. For example, if $p = 3$ then

$$U(x) = \frac{\sqrt{2}}{\cosh(x)}.$$

In the case $N \geq 2$ we need to use the functional framework outlined before. From $(N.2)$, $\mathcal{J}$ is bounded from below on $\mathcal{N}$. By the Ekeland variational principle, there exist sequences $u_k \in \mathcal{N}$, $\mu_k \in \mathbb{R}$ such that

$$\mathcal{J}(u_k) \to c := \inf\{\mathcal{J}(u) : u \in \mathcal{N}\} > 0, \quad \mathcal{J}'(u_k) - \mu_k\mathcal{G}'(u_k) \to 0. \tag{13.8}$$

Since we can substitute $u_k$ with $|u_k|$, we can assume that $u_k \geq 0$. Moreover, if $u_k^*$ denotes the Schwarz symmetric function associated to $u_k$,[1] let $t_k^* > 0$ be such that $t_k^* u_k^* \in \mathcal{N}$. It is well known that $\|u_k^*\|^2 \leq \|u_k\|^2$, while $\int |u_k^*|^{p+1} = \int |u_k|^{p+1}$. Then (13.6) yields

$$t_k^* = \frac{\|u_k^*\|^2}{\int |u_k^*|^{p+1}} \leq \frac{\|u_k\|^2}{\int |u_k|^{p+1}} = 1.$$

Moreover,

$$\mathcal{J}(t_k^* u_k^*) = (\tfrac{1}{2} - \tfrac{1}{p+1})(t_k^*)^2 \|u_k^*\|^2 \leq (\tfrac{1}{2} - \tfrac{1}{p+1})(\|u_k\|^2 = \mathcal{J}(u_k).$$

Therefore, we can also suppose that $u_k$ is radial.

From the first expression of (13.8) and (13.5) it follows that $(\frac{1}{2} - \frac{1}{p+1})\|u_k\|^2 \to c$ and hence there exists $c' > 0$ such that $\|u_k\| \leq c'$. Without relabeling, we can assume that $u_k \rightharpoonup u$, weakly in $E$. Since $u_k$ are radially symmetric, and the subspace of the radially symmetric functions in $E$ is compactly embedded in $L^{p+1}(\mathbb{R}^N)$, $N \geq 2$, we deduce that there exists $U \in E$ such that $u_k \to U$ strongly in $L^{p+1}(\mathbb{R}^N)$. From the second expression of (13.8) we get

$$(\mathcal{J}'(u_k) \mid u_k) - \mu_k (\mathcal{G}'(u_k) \mid u_k) \to 0.$$

Since $(\mathcal{J}'(u_k) \mid u_k) = \mathcal{G}(u_k) = 0$ and $(\mathcal{G}'(u_k) \mid u_k) \leq (1-p)r^2 < 0$, see $(N.3)$, it follows that $\mu_k \to 0$. Setting $h(u) = \frac{1}{p+1} \int |u|^{p+1}$, one has $\mathcal{J}'(u_k) = u_k - h'(u_k)$ and therefore the second expression of (13.8) yields $u_k = h'(u_k) - \mu_k \mathcal{G}'(u_k) + o(1)$. Then we find $u_k \to U (= h'(U))$ strongly in $E$. It follows immediately that $U \in \mathcal{N}$, $\mathcal{J}(U) = c$ and $\mathcal{J}'(U) \perp \mathcal{N}$. Using Lemma 13.1.1, we deduce that $U$ is a nontrivial solution of (13.3). Since $u_k$ are radially symmetric and non-negative, it follows that $U$ is radially symmetric and $U \geq 0$. Finally, from the equation $-\Delta U + U = U^p$, the fact that $U \not\equiv 0$ and the maximum principle, we deduce that $U > 0$. ☐

Our next result deals with the nonautonomous equation

$$-\Delta u + q(x)u = |u|^{p-1}u, \quad u \in H^1(\mathbb{R}^N), \tag{13.9}$$

where $N > 2$, $1 < p < 2^* - 1$. We assume that the potential $q \in C(\mathbb{R}^N)$ satisfies

$(q1)$ $\exists q_0 > 0$ such that $q(x) > q_0$, for every $x \in \mathbb{R}^N$,
$(q2)$ $\lim_{|x| \to +\infty} q(x) = +\infty$.

[1] For the definition and properties of the Schwarz symmetrization, see [59].

We follow [78]. Solutions of (13.9) are the stationary points of the functional

$$\mathcal{J}(u) = \frac{1}{2}\int \left[|\nabla u|^2 + q\, u^2\right] - \frac{1}{p+1}\int |u|^{p+1}$$

on $H^1(\mathbb{R}^N)$. Actually we will work on

$$\mathcal{E} = \{u \in H^1(\mathbb{R}^N) : \int \left[|\nabla u|^2 + q\, u^2\right] < +\infty\}.$$

By $(q1)$, the space $\mathcal{E}$ is endowed with the norm

$$\|u\|_{\mathcal{E}}^2 = \int \left[|\nabla u|^2 + q\, u^2\right].$$

With this notation we can write

$$\mathcal{J}(u) = \frac{1}{2}\|u\|_{\mathcal{E}}^2 - \frac{1}{p+1}\int |u|^{p+1}$$

It is easy to check that $\mathcal{E} \subset H^1(\mathbb{R}^N) \subset L^{p+1}(\mathbb{R}^N)$ with continuous embedding (see Exercise 9).

**Theorem 13.1.3** *If* $(q1)$–$(q2)$ *hold then* (13.9) *has a positive (and a negative) solution.*

*Proof* It is clear that $\mathcal{J}$ has the mountain pass geometry. As usual, we let

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}(\gamma(t))$$

denote the mountain pass value. One has that $c > 0$. To apply the mountain pass theorem we should check the $(PS)$ condition. Unfortunately, we cannot carry out the procedure used in the case of problems on a bounded domain, because the embedding $\mathcal{E} \subset L^p$ fails to be compact. In the previous theorem this difficulty has been bypassed using the fact that the problem was autonomous and this allowed us to work with radial functions. Here we will use the fact that the potential $q$ satisfies $(q2)$.

First of all, the Ekeland variational principle (Theorem 5.4.2) yields a sequence $u_n \in \mathcal{E}$ such that

$$\mathcal{J}(u_n) \to c, \quad \mathcal{J}'(u_n) \to 0. \tag{13.10}$$

Standard arguments imply that $\|u\|_{\mathcal{E}} \le M$. Hence, up to a subsequence, $u_n$ converges weakly in $\mathcal{E}$, and strongly in $L^{p+1}_{loc}(\mathbb{R}^N)$, to some $u \in \mathcal{E}$. Moreover, from (13.10) it follows that

$$(\mathcal{J}'(u_n)|v)_{\mathcal{E}} \longrightarrow (\mathcal{J}'(u)|v)_{\mathcal{E}}. \tag{13.11}$$

Then $u$ is a weak (and by regularity, strong) solution of (13.9). Let us show that $u \neq 0$.

From (13.11) we infer that, for $n \gg 1$,

$$\frac{1}{2}c \le \mathcal{J}(u_n) - \frac{1}{2}\mathcal{J}'(u_n) = \int \left[\frac{1}{2} - \frac{1}{p+1}\right] |u_n|^{p+1}. \tag{13.12}$$

Inserting into (13.12) the Gagliardo–Nirenberg interpolation inequality (see [36, p. 23])

$$\|u_n\|_{p+1} \le C_1 \|\nabla u_n\|_2^{\theta} \|u_n\|_2^{1-\theta}, \quad \text{with } \theta = N\left[\frac{1}{2} - \frac{1}{p+1}\right]$$

we find that

$$\frac{1}{2}c \le C_2 \left(\|\nabla u_n\|_2^{(p+1)(\theta)} \|u_n\|_2^{(p+1)(1-\theta)}\right),$$

where $C_2 = C_1\left[\frac{1}{2} - \frac{1}{p+1}\right]$. This inequality and the fact that $\|u_n\|_{\mathcal{E}}$ is bounded implies that there exists $c' > 0$ such that

$$c' \le \|u_n\|_2^2. \tag{13.13}$$

Letting $B_R = \{x \in \mathbb{R}^N : |x| < R\}$, we can write, for any $R > 0$,

$$\int |u_n|^2 = \int_{B_R} |u_n|^2 + \int_{\mathbb{R}^N \setminus B_R} |u_n|^2. \tag{13.14}$$

The last integral can be estimated as follows:

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R} |u_n|^2 &\le \frac{1}{\inf_{\mathbb{R}^N \setminus B_R} q^2} \int_{\mathbb{R}^N \setminus B_R} |q\, u_n|^2 \\ &\le \frac{1}{\inf_{\mathbb{R}^N \setminus B_R} q^2} \|u_n\|_{\mathcal{E}}^2 \\ &\le \frac{1}{\inf_{\mathbb{R}^N \setminus B_R} q^2} M^2. \end{aligned}$$

From (13.13) and (13.14) we get

$$c' \le \int_{B_R} |u_n|^2 + \frac{1}{\inf_{\mathbb{R}^N \setminus B_R} q^2} M^2.$$

Using ($q2$) we find

$$\lim_{R \to \infty} \frac{1}{\inf_{\mathbb{R}^N \setminus B_R} q^2} = 0,$$

and thus there exists $R_0 > 0$ such that for all $R > R_0$ and $n \gg 1$ one has

$$\frac{1}{2}c' \le \int_{B_R} |u_n|^2.$$

With fixed $R > R_0$, since $u_n$ converges strongly to $u$ in $L^2(B_R)$, we obtain that

$$\int_{B_R} |u|^2 = \lim_{n \to \infty} \int_{B_R} |u_n|^2 \ge \frac{1}{2}c' > 0.$$

This implies that $u \not\equiv 0$. Finally, substituting the nonlinearity $|u|^{p-1}u$ with its positive part, we find that $u > 0$. □

## 13.2 Semiclassical States of NLS Equations with Potentials

In order to study the relationship between classical and quantum mechanics, one introduces a small parameter $\varepsilon \in \mathbb{R}$ and considers the problem

$$-\varepsilon^2 \Delta u + V(x)u = u^p, \tag{13.15}$$

where from this point on $2 < p + 1 < 2^*$. We want to see if (13.15) has a positive solution for $\varepsilon \sim 0$, $u_\varepsilon$ which concentrates at some point $x^*$, in the sense that

$$\forall \delta > 0,\ \exists \varepsilon^* > 0,\ R > 0\ :\ u_\varepsilon(x) \le \delta,\ \forall |x - x^*| \le \varepsilon R,\ \varepsilon < \varepsilon^*.$$

These solutions look like a soliton whose energy concentrates at $x^*$ and are called semiclassical states of (13.15). We anticipate that the concentration point $x^*$ is a stationary point of $V$.

There is a broad literature on the existence of semiclassical states, starting with the paper by Floer and Weinstein [52]. In addition to solutions concentrating at a single point, the existence of semiclassical states with many (possibly infinitely many) peaks has been proved, in dependence of suitable properties of the potential $V$ and of the nature of its stationary points.

Here we will limit ourselves to discussing a basic result, following [8].

We will suppose that $V$ satisfies

$(V1)$ $0 < \inf_{\mathbb{R}^N} V(x) < \sup_{\mathbb{R}^N} V(x) < +\infty$,
$(V2)$ $V$ has a non-degenerate stationary point at $x^*$: there exists $\alpha > 0$ such that $V(x) - V(x^*) = \pm\alpha|x - x^*|^2 + o(|x - x^*|^2)$.

Up to a translation we can assume that

$$x^* = 0, \quad \text{and} \quad V(0) = 1.$$

To highlight that (13.15) is perturbation in nature it is convenient to perform the change of variable $x \mapsto \varepsilon x$. Then (13.15) becomes

$$-\Delta u + V(\varepsilon x)u = u^p,$$

or else

$$-\Delta u + u + (V(\varepsilon x) - 1)u = u^p. \tag{13.16}$$

Clearly, if $u$ is a solution of (13.16) then $u(x/\varepsilon)$ is a semiclassical state of (13.15) concentrating at $x^* = 0$.

We will seek solutions of (13.16) as critical points of the functional

$$\mathcal{I}_\varepsilon(u) = \mathcal{I}(u) + \mathcal{G}_\varepsilon(u), \quad u \in E := H^1(\mathbb{R}^N),$$

where

$$\mathcal{I}(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int |u|^{p+1}$$

and

$$\mathcal{G}_\varepsilon(u) = \tfrac{1}{2}\int (V(\varepsilon x) - 1)u^2.$$

We will find critical points of $\mathcal{I}_\varepsilon$ by means of the perturbation results discussed in Sect. 5.6. We will keep the notation introduced therein. In particular, we will continue to consider the case in which $Z$ is one dimensional, with parameter $s \in \mathbb{R}$.

**Lemma 13.2.1** *Assumptions* (A1), (A2) *and* (A3) *of Sect. 5.6 are verified.*

*Proof* One has

$$\mathcal{G}'_\varepsilon(z)[v] = \int (V(\varepsilon x) - 1)zv \le \left[\int (V(\varepsilon x) - 1)^2 z^2\right]^{1/2} \|v\|_{L^2}.$$

Moreover, $(V2)$ yields $V(\varepsilon x) - 1 = \alpha^2\varepsilon^2 x^2 + o(\varepsilon^2 x^2)$. On the other hand, since $z$ has an exponential decay at infinity, one finds that $\int x^2 z^2 \le c_1$. As a consequence,

$$\left[\int (V(\varepsilon x) - 1)^2 z^2\right]^{\frac{1}{2}} \le c_2\alpha^2\,\varepsilon^2 + o(\varepsilon^2), \tag{13.17}$$

and $(A1)$ follows.

To prove $(A2)$ let $W$ be the space $\langle z'\rangle^\perp$ and write $W = \langle z\rangle \bigoplus W'$. It is well known that, for $\varepsilon = 0$,

$$P\mathcal{I}''(z)[z,z] > 0, \quad \forall s \in \mathbb{R}.$$

It follows that there exists $c > 0$ such that

$$P\mathcal{I}''_\varepsilon(z)[z,z] \ge c, \quad \varepsilon \sim 0.$$

Moreover, it is easy to see that, taking $c$ possibly different,

$$P\mathcal{I}''_\varepsilon(z)[v,v] \le -c, \quad \forall\, v \in W', \quad \varepsilon \sim 0.$$

Then $P\mathcal{I}''_\varepsilon(z)$ is invertible, provided $\varepsilon$ is sufficiently small and $(A2)$ holds.

Finally the proof of $(A3)$ is trivial. □

*Remark 13.2.2* Since $\|w_\varepsilon\| \le c_1\|\mathcal{G}'_\varepsilon(z)\|$, (13.17) implies that $\|w_\varepsilon\| = O(\varepsilon^2)$.

We are now in position to state the following result, which follows immediately from the preceding lemmas and from the perturbation Theorem 5.6.5.

**Theorem 13.2.3** *Let* $(V1)$–$(V2)$ *hold. Then for* $\varepsilon$ *small,* (13.15) *has a semiclassical state which concentrates at* $x^*$.

*Proof* In order to apply the perturbation Theorem 5.6.5, it remains to show that the reduced functional

$$\widetilde{\mathcal{I}}_\varepsilon(s) = \mathcal{I}_\varepsilon(z(s) + w_\varepsilon(s)), \quad s \in \mathbb{R}$$

has a stationary point $s_\varepsilon^*$. One has

$$\begin{aligned}\mathcal{I}_\varepsilon(z(s) + w_\varepsilon(s)) &= \mathcal{I}(z(s) + w_\varepsilon(s)) + \mathcal{G}_\varepsilon(z(s) + w_\varepsilon(s))\\ &= \mathcal{I}(z(s)) + O(\|w_\varepsilon(s)\|^2) + \tfrac{1}{2}\int (V(\varepsilon x) - 1)(z(s) + w_\varepsilon(s))^2.\end{aligned}$$

Let us remark that $z(s) = U(\,\cdot - s)$ and then $\mathcal{I}(z(s)) = c_0$. Moreover, using Remark 13.2.2, we infer

$$\mathcal{I}_\varepsilon(z(s) + w_\varepsilon(s)) = c_0 + \tfrac{1}{2}\int (V(\varepsilon x) - 1)z^2(s) + o(\varepsilon^2).$$

Finally, $(V2)$ implies

$$\int (V(\varepsilon x) - 1)z^2(s) = \pm\alpha\varepsilon^2 \int x^2 U^2(x - s) = \pm\alpha\varepsilon^2 \int (y + s)^2 U^2(y)dy.$$

Since $U$ is an even function, $\int ysU(y)dy = 0$ and thus

$$\int (V(\varepsilon x) - 1)z^2(s) = \pm\alpha\, c_1\varepsilon^2 s^2 + c_2,$$

where

$$c_1 = \tfrac{1}{2}\int U^2(y)dy, \quad c_2 = \tfrac{1}{2}\int y^2 U^2(y)dy.$$

In conclusion we find that

$$\widetilde{\mathcal{I}}_\varepsilon(s) = \mathcal{I}_\varepsilon(z(s) + w_\varepsilon(s)) = c_3 \pm \alpha\, c_1\varepsilon^2 s^2 + o(\varepsilon^2).$$

Hence the reduced functional $\widetilde{\mathcal{I}}_\varepsilon$ has a stationary point $s_\varepsilon^*$ such that $s_\varepsilon^* \to 0$ as $\varepsilon \to 0$. The result now follows from Theorem 5.6.5. □

*Remark 13.2.4* In the case discussed above the presence of a perturbation like $\mathcal{G}_\varepsilon(u)$ does not allow us to use the implicit function theorem to solve the auxiliary equation as in the case $\varepsilon\mathcal{G}(u)$. Actually, $P\mathcal{I}'_\varepsilon$ could fail to be $C^1$, because $\mathcal{G}''_\varepsilon(u)$ might not tend to zero as $\varepsilon \to 0$. The difficulty is overcome by using $(A1)$–$(A3)$. A specific example is reported in Exercise 51.

*Remark 13.2.5* There is a great deal of work on problems like (13.15) under several different assumptions on the potential $V$. For instance, different approaches to find semiclassical states can be found in [45] or [86].

## 13.3 Systems of NLS Equations

In this section we will study a system of linearly coupled NLS equations, such as

$$\begin{cases} -u'' + u = u^3 + \lambda v, & x \in \mathbb{R}, \\ -v'' + v = v^3 + \lambda u, & x \in \mathbb{R}. \end{cases} \tag{13.18}$$

These systems typically arise in nonlinear optics. We follow [7].

First of all, in addition to the trivial solution $(0,0)$, there are two families of nontrivial solution pairs. First, if we look for solutions such that $u = v$, we find the equation

$$-u'' + (1-\lambda)u = u^3,$$

whose solution is

$$U_{1-\lambda}(x) = \frac{\sqrt{2(1-\lambda)}}{\cosh\left(\sqrt{(1-\lambda)}x\right)}, \quad 0 \le \lambda \le 1. \tag{13.19}$$

On the other hand, if we look for solutions such that $u = -v$, we find

$$-u'' + (1+\lambda)u = u^3,$$

whose solution is

$$U_{1+\lambda}(x) = \frac{\sqrt{2(1+\lambda)}}{\cosh\left(\sqrt{(1+\lambda)}x\right)}, \quad \lambda \ge 0.$$

Hence (13.18) has the following two families of nontrivial solutions:

- $(U_{1-\lambda}, U_{1-\lambda}),\ 0 \le \lambda \le 1$ (symmetric states);
- $(U_{1+\lambda}, -U_{1+\lambda}),\ \lambda \ge 0$ (anti-symmetric states).

We now look for solutions of (13.18) different from the symmetric and anti-symmetric states. Let us start with the case in which the parameter $\lambda > 0$ is small. We set

- $X = \{u \in C^2(\mathbb{R}) : u(x) = u(-x),\ \lim_{|x|\to\infty} u(x) = 0\}$,
- $\mathbb{X} = X \times X$,
- $Y = \{u \in C(\mathbb{R}) : u(x) = u(-x)\}$,
- $\mathbb{Y} = Y \times Y$.

Let us point out that for $\lambda = 0$, (13.18) has the following nontrivial solutions: $(U,0)$, $(0,U), (U,\pm U)$.

**Theorem 13.3.1** *From each $(U,0), (0,U), (U,\pm U)$ there branches off, for $\lambda > 0$ small enough, a unique curve $(u_\lambda, v_\lambda) \in \mathbb{X}$ of solutions of (13.18), such that $u_\lambda \not\equiv 0$, $v_\lambda \not\equiv 0$.*

*Proof* Consider the map $F : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$

$$F(\lambda; u, v) = (-u'' + u - u^3 - \lambda v, -v'' + v - v^3 - \lambda u).$$

In order to apply the implicit function theorem at $\lambda = 0$, $(u, v) = (U, 0)$, let us consider the partial derivative $d_{(u,v)}F(0; U, 0)$,

$$d_{(u,v)}F(0; U, 0)[u, v] = (-u'' + u - 3U^2u, -v'' + v)$$

and the equation $d_{(u,v)}F(0; U, 0)[u, v] = (h, k) \in \mathbb{Y}$, namely the decoupled system

$$\begin{cases} -u'' + u - 3U^2u = h, \\ \qquad\quad -v'' + v = k. \end{cases}$$

We claim that it has a unique solution. This is trivially true for the latter equation. As for the former, we first remark that $u = U'$ is a nontrivial solution of the linearized equation

$$-u'' + u - 3U^2u = 0,$$

and the only one satisfying $\lim_{|x|\to\infty} u(x) = 0$ (see Appendix 13.5 at the end of this chapter). Since $U'$ is an odd function, the equation $-u'' + u - 3U^2u = 0$, $u \in X$, has only the trivial solution, and the claim follows.

The same arguments hold for the derivative

$$d_{(u,v)}F(0; U, \pm U)[u, v] = (-u'' + u - 3U^2u, -v'' + v - 3U^2v).$$

These arguments allow us to apply the implicit function Theorem 3.2.1 to $F(\lambda; u, v) = 0$, and the existence of the families $(u_\lambda, v_\lambda)$ follows. Of course, none of the components can be identically zero because if $\lambda > 0$, (13.18) has no solution of the form $(u, 0)$ or $(0, v)$. □

Next, we look for secondary bifurcations from the family of symmetric states $(U_{1-\lambda}, U_{1-\lambda})$. We will use Theorem 6.1.3, concerned with the bifurcation from the simple eigenvalue. For this, we change the variable, setting

$$\mathbf{w} = (w_1, w_2), \quad w_1 = u - U_{1-\lambda}, \quad w_2 = v - U_{1-\lambda},$$

and consider the map

$$\mathbb{F}(\lambda; \mathbf{w}) = F(\lambda, w_1 + U_{1-\lambda}, w_2 + U_{1-\lambda}),$$

in such a way that (we set $\mathbf{0} = (0, 0)$)

$$\mathbb{F}(\lambda; \mathbf{0}) = F(\lambda, U_{1-\lambda}, U_{1-\lambda}) \equiv \mathbf{0}.$$

We need to study the operator

$$\mathbb{T}_\lambda := d_{\mathbf{w}}\mathbb{F}(\lambda; \mathbf{0}) = d_{(u,v)}F(\lambda; U_{1-\lambda}, U_{1-\lambda}) \in L(\mathbb{X}, \mathbb{Y}).$$

Let us start with Ker $[\mathbb{T}_\lambda]$ and consider the linearized system $\mathbb{T}_\lambda[u, v] = (0, 0)$:

$$\begin{cases} -u'' + u - 3U_{1-\lambda}^2 u - \lambda v = 0, \\ -v'' + v - 3U_{1-\lambda}^2 v - \lambda u = 0, \end{cases} \tag{13.20}$$

with $(u, v) \in \mathbb{X}$. Setting

$$\begin{cases} \phi = u + v, \\ \psi = u - v, \end{cases}$$

system (13.20) becomes

$$\begin{cases} -\phi'' + (1-\lambda)\phi - 3U_{1-\lambda}^2 \phi = 0, \\ -\psi'' + (1+\lambda)\psi - 3U_{1-\lambda}^2 \psi = 0. \end{cases} \tag{13.21}$$

Since (13.21) is decoupled, we can study the two equations separately. As before, the unique nontrivial solution of the first equation is $U'_{1-\lambda}$, which does not belong to $X$, whence $\phi = 0$. Let us now consider the second equation in (13.21). It is of the type (13.30) discussed in the first item of Appendix 13.5, namely

$$-\psi'' + Q_\lambda(x)\psi = 0,$$

with

$$Q_\lambda(x) = 1 + \lambda - 3U_{1-\lambda}^2,$$

and

$$b_\lambda = \lim_{|x|\to\infty} Q_\lambda(x) = 1 + \lambda.$$

Let us denote by $A_\lambda$ the operator

$$A_\lambda(\psi) := -\psi'' + Q_\lambda(x)\psi, \quad \psi \in H^1(\mathbb{R}).$$

**Lemma 13.3.2** *For $\lambda \in [0, 1)$ the first two eigenvalues $\nu_1(\lambda) < \nu_2(\lambda)$ of $A_\lambda$ are given by:*

(*a*) $\nu_1(\lambda) = 5\lambda - 3$,
(*b*) $\nu_2(\lambda) = 2\lambda$.

*Proof* Let us compute $A_\lambda(f_\lambda)$ with

$$f_\lambda(x) := \frac{1}{\cosh^2(\sqrt{1-\lambda}\, x)}$$

With this notation, one has $Q_\lambda(x) = 1 + \lambda - 6(1-\lambda) f_\lambda$. Since

$$f_\lambda'' = -2(1-\lambda) f_\lambda + 6(1-\lambda)\sinh^2(\sqrt{1-\lambda}\, x) f_\lambda^2,$$

and $Q_\lambda f_\lambda = (1+\lambda) f_\lambda - 6(1-\lambda) f_\lambda^2$, we find (the dependence of $f$ on $\lambda$ is understood)

$$\begin{aligned} A_\lambda(f) &= (3-\lambda) f - 6(1-\lambda)\sinh^2(\sqrt{1-\lambda}\,x) f^2 - 6(1-\lambda) f^2 \\ &= (3-\lambda) f - 6(1-\lambda)(\sinh^2(\sqrt{1-\lambda}\,x) + 1) f^2 \\ &= (3-\lambda) f - 6(1-\lambda)\cosh^2(\sqrt{1-\lambda}\,x) f^2 \\ &= (3-\lambda) f - 6(1-\lambda) f = (5\lambda - 3) f. \end{aligned}$$

Since $f_\lambda > 0$, it follows (see Appendix 13.5-(1)) that $5\lambda - 3$ is the first eigenvalue of $A_\lambda$, proving $(a)$.

Next, let us set $g_\lambda = U'_{1-\lambda}$ and remark that

$$-g''_\lambda + (1-\lambda) g_\lambda - 3U^2_{1-\lambda} g_\lambda = 0.$$

Then one finds:

$$A_\lambda(g_\lambda) = -g''_\lambda + (1+\lambda) g_\lambda - 3U^2_{1-\lambda} g_\lambda = 2\lambda g_\lambda \,.$$

Since $g_\lambda$ has a single zero, it follows that $2\lambda$ is the second eigenvalue of $A_\lambda$, proving $(b)$. Let us point out that $\nu_1(\lambda) < \nu_2(\lambda) < b_\lambda$ provided $\lambda < 1$. □

From the previous lemma we deduce the following.

**Lemma 13.3.3** (i) *For all* $\lambda \in [0,1)$, $\lambda \neq 3/5$, *the operator* $\mathbb{T}_\lambda$ *is invertible.*
(ii) *For* $\lambda = 3/5$, Ker $[\mathbb{T}_\lambda]$ *is one dimensional and spanned by* $\varphi^* = (f^*, f^*)$, *where* $f^* = f_{3/5}$. *Moreover, the* Range$[\mathbb{T}_\lambda]$ *equals the subspace* $\mathbb{Y}_0 \subset \mathbb{Y}$,

$$\mathbb{Y}_0 = \{(h,k) \in \mathbb{Y} : \int h f^* = \int k f^* = 0\}.$$

*Proof* (i-1) For $0 \le \lambda < 3/5$, resp. $3/5 < \lambda < 1$, one has that $\nu_1(\lambda) < 0 < \nu_2(\lambda)$, resp. $\nu_1(\lambda) > 0$, and therefore Ker $[A_\lambda] = \{0\}$. Then the solution of (13.21) is given by $\phi = 0$, $\psi = 0$. Since

$$\begin{aligned} u &= \tfrac{1}{2}(\phi + \psi), \\ v &= \tfrac{1}{2}(\phi - \psi), \end{aligned} \tag{13.22}$$

it follows that (13.20) has only the trivial solution $(0,0)$, proving that Ker $[\mathbb{T}_\lambda] = \{(0,0)\}$ for $\lambda \in [0,3/5) \cup (3/5,1)$.

(i-2) To show that the system $\mathbb{T}_\lambda[u,v] = (h,k)$ has a unique solution for any $(h,k) \in \mathbb{Y}$ and for $\lambda \in [0,3/5) \cup (3/5,1)$, it suffices to pass to the decoupled system

$$\begin{aligned} -\phi'' + (1-\lambda)\phi - 3U^2_{1-\lambda}\phi &= h + k, \\ -\psi'' + (1+\lambda)\psi - 3U^2_{1-\lambda}\psi &= h - k, \end{aligned} \tag{13.23}$$

which has a unique solution. From (i-1)–(i-2) it follows that (i) holds.

(ii) For $\lambda = 3/5$, Lemma 13.3.2 implies that $\mathrm{Ker}\,[A_\lambda]$ is spanned by $f^*$. Using again (13.22), it follows that $\mathrm{Ker}\,[\mathbb{T}_\lambda]$, $\lambda = 3/5$, is spanned by $\varphi^* = (f^*, f^*)$. Moreover, arguing as before, the system $\mathbb{T}_\lambda[u, v] = (h, k)$ is equivalent to (13.23), which has, for $\lambda = 3/5$, a unique solution, provided

$$\int (h - k) f^* = 0. \tag{13.24}$$

Furthermore, multiplying the system $\mathbb{T}_\lambda[u, v] = (h, k)$ by $f^*$ and integrating by parts, it follows that

$$\int u\left[-(f^*)'' + f^* - 3U_{1-\lambda}^2 f^*\right] - \lambda \int v f^* = \int h f^*,$$

$$\int v\left[-(f^*)'' + f^* - 3U_{1-\lambda}^2 f^*\right] - \lambda \int u f^* = \int k f^*.$$

From $A_\lambda(f^*) = 0$, $\lambda = 3/5$, we infer $-(f^*)'' + f^* - 3U_{1-3/5}^2 f^* = \frac{3}{5} f^*$ and hence

$$\tfrac{3}{5} \int u f^* - \tfrac{3}{5} \int v f^* = \int h f^*,$$

$$\tfrac{3}{5} \int v f^* - \tfrac{3}{5} \int u f^* = \int k f^*.$$

This and (13.24) imply $\int h f^* = \int k f^* = 0$. This shows that for $\lambda = 3/5$, $\mathrm{Range}[\mathbb{T}_\lambda] = \mathbb{Y}_0$, and completes the proof of the lemma. □

**Theorem 13.3.4** *For $\lambda^* = 3/5$ there is a branching of solutions $(u_\lambda, v_\lambda) \in \mathbb{X}$ of (13.18) from the family of symmetric states $(U_{1-\lambda}, U_{1-\lambda})$, and it is the only one (see Fig. 13.1).*

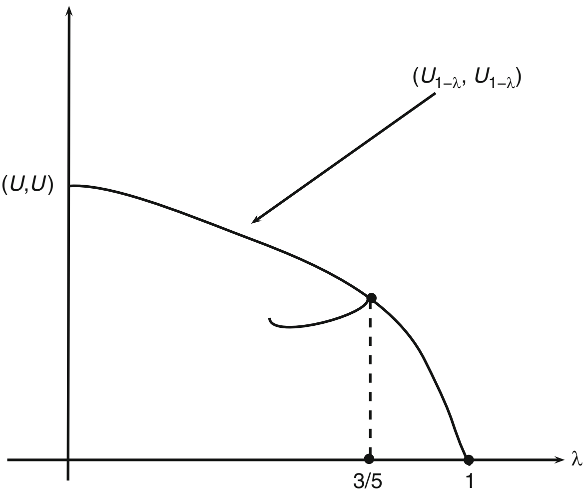



**Fig. 13.1** Bifurcation diagram for Theorem 13.3.4

*Proof* As anticipated before, we will use Theorem 6.1.3. Lemma 13.3.3-(ii) shows that assumptions $(F.1)$ and $(F.2)$ of Sect. 6.1 hold, with $\mu = \lambda^*$ and $\varphi = \varphi^*$. As for condition $(F.3)$, let us evaluate the mixed derivative

$$d_{\lambda,\mathbf{w}}\mathbb{F}(\lambda,0)[u,v] = (-3d_\lambda U_{1-\lambda}^2 u - v, -3d_\lambda U_{1-\lambda}^2 v - u). \tag{13.25}$$

In the Appendix 13.5-(2) it is proved that

$$d_{\lambda,\mathbf{w}}\mathbb{F}(\lambda^*,0)[u,v] = (h^*, h^*),$$

for some $h^* \in Y$ such that

$$\int h^* f^* < 0. \tag{13.26}$$

It follows that $d_{\lambda,\mathbf{w}}\mathbb{F}(\lambda^*,0)[\varphi^*] \notin \mathbb{Y}_0$, namely that $(F.3)$ holds. Then an application of Theorem 6.1.3 implies that $\lambda = \lambda^*$ is a bifurcation point.

Moreover, Lemma 13.3.3-(i) implies that any $\lambda \in (0,1)$, $\lambda \neq 3/5$, cannot be a bifurcation point. □

*Remark 13.3.5* System (13.18) has other solutions, different in nature from the ones found before. For example, the authors have proved in [12] the existence, for $\lambda = \varepsilon$ small, of solutions $(u_\varepsilon, v_\varepsilon)$ such that, as $\varepsilon \to 0$,

$$u_\varepsilon(x) \sim U(x+\xi_\varepsilon) + U(x-\xi_\varepsilon), \quad v_\varepsilon(x) \sim -U(x),$$

where $\xi_\varepsilon \sim \log(1/\varepsilon)$. The authors suspect that these solutions can be continued for $\lambda \in [0,1)$ and will converge, as $\lambda \to 1$ to the anti-symmetric pair $(U_2, -U_2)$. Let us mention that [12] also deals with the PDE counterpart of (13.18) in dimension $n = 2, 3$. In such a case it is proved that there exist solutions whose first component has many bumps located near the vertices of any regular polygon with less than six sides, resp. any regular polyhedra but the dodecahedron, in dimension $n = 2$, resp. $n = 3$.

*Remark 13.3.6* There is a numerical evidence that a secondary bifurcation branches off at $\lambda = 1$ from the anti-symmetric states $(U_{1+\lambda}, -U_{1+\lambda})$. However, we do not know a rigorous proof of this result.

*Remark 13.3.7* For other results dealing with nonlinearly coupled NLS equations we refer to [11].

## 13.4 Nonautonomous Systems

Here we discuss the results of [7] dealing with the nonautonomous system

$$\begin{aligned} -u'' + u &= (1+\varepsilon b_1(x))u^3 + \lambda v, \quad x \in \mathbb{R}, \\ -v'' + v &= (1+\varepsilon b_2(x))v^3 + \lambda u, \quad x \in \mathbb{R}, \end{aligned} \tag{13.27}$$

where $b_i,\ i = 1, 2$ satisfy

$$b_i \in L^\infty(\mathbb{R}), \qquad \lim_{|x|\to\infty} b_i(x) = 0,\ i = 1, 2. \tag{13.28}$$

Solutions of (13.27) will be searched as critical points of

$$\mathcal{I}_{\varepsilon,\lambda}(u, v) = \mathcal{I}_\lambda(u) + \mathcal{I}_\lambda(v) - \varepsilon\mathcal{G}(u, v), \quad (u, v) \in \mathbb{E}$$

where $\mathbb{E} = H^1(\mathbb{R}) \times H^1(\mathbb{R})$,

$$\mathcal{I}_\lambda(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{4}\int u^4 - \tfrac{\lambda}{2}\int u^2$$

and

$$\mathcal{G}(u, v) = \tfrac{1}{4}\int \left(b_1(x))u^4 + b_2(x)v^4\right).$$

For $\varepsilon = 0$ the unperturbed functional $\mathcal{I}_{\varepsilon,\lambda}$ has the following manifold of critical points:

$$Z_\lambda = \{z_\xi = (U_{1-\lambda}(x+\xi), U_{1-\lambda}(x+\xi)),\ \xi \in \mathbb{R},\ 0 < \lambda < 1\}.$$

In order to use the perturbation methods studied in Sect. 5.6, we will check that the assumptions (A1)–(A3) hold.

Set $\mathbf{w} = (w_1, w_2)$. From

$$(\mathcal{G}'(z_\xi)|\mathbf{w}) = \int (b_1(x)w_1 + b_2(x)w_2)U^3_{1-\lambda}(x+\xi),$$

and using (13.28), an argument already used before shows that $\|\mathcal{G}'(z_\xi)\| \le c$, for some $c > 0$, proving (A1).

Condition (A3) is also easily verified. To prove that (A2) holds we will show that $Z_\lambda$ is non-degenerate, in the sense that $T_z Z_\lambda = \mathrm{Ker}\,[\mathcal{I}''_\lambda(z)]$ for all $z \in Z$. Up to translation, it suffices to take $\xi = 0$. Let

$$\Gamma = \left\{\lambda \in \mathbb{R} : 0 < \lambda < 1,\ \lambda \neq \frac{3}{5}\right\}.$$

**Lemma 13.4.1** *If $\lambda \in \Gamma$, then the kernel of $\mathcal{I}''_\lambda[(U_{1-\lambda}, U_{1-\lambda})]$ is spanned by $(U'_{1-\lambda}, U'_{1-\lambda})$.*

*Proof* We have to prove that any $\mathbf{w} = (w_1, w_2)$ solving the linear system

$$\begin{aligned} -w_1'' + w_1 - 3U^2_{1-\lambda}w_1 - \lambda w_2 = 0,\\ -w_2'' + w_2 - 3U^2_{1-\lambda}w_2 - \lambda w_1 = 0, \end{aligned} \tag{13.29}$$

has the form $\mathbf{w} = (U'_{1-\lambda}, U'_{1-\lambda})$. Setting

$$\psi = w_1 - w_2,$$

the function $\psi$ solves

$$A_\lambda(\psi) = -\psi'' + Q_\lambda(x)\psi = \lambda\psi, \quad \psi \in H^1(\mathbb{R}), \tag{13.30}$$

where

$$Q_\lambda(x) = 1 + \lambda - 3U^2_{1-\lambda}(x) = 1 + \lambda - \frac{6(1-\lambda)}{\cosh^2(\sqrt{1-\lambda}\,x)}.$$

Let us show that if $\lambda \in \Gamma$ then $\psi = 0$. By Lemma 13.3.2-(a), the first eigenvalue of (13.30) is

$$\Lambda_\lambda = \inf\left\{\int [(u')^2 + Q_\lambda(x)u^2] : \int u^2 = 1\right\} = 5\lambda - 3.$$

Since for $\lambda > \frac{3}{5}$ one has that $\Lambda_\lambda > 0$, it follows that $\lambda = 0$ is not an eigenvalue of (13.30). Therefore $\psi = 0$ provided $\frac{3}{5} < \lambda < 1$.

We next deal with the case $0 < \lambda < \frac{3}{5}$. In this case, by Lemma 13.3.2, it holds that $5\lambda - 3 < 0 < 2\lambda$ with $2\lambda$ is the second eigenvalue (and $5\lambda - 3$ the first one) of (13.30). Thus $\lambda = 0$ is not an eigenvalue of (13.30), proving that $\psi = 0$.

From $w_1 = w_2$, we find that $w_2$ satisfies

$$-w_2'' + (1-\lambda)w_2 - 3U^2_{1-\lambda}w_2 = 0,$$

and hence $w_2 = U'_{1-\lambda}$, completing the proof. □

**Lemma 13.4.2** *If $\lambda \in \Gamma$, then $P\mathcal{I}''_\lambda(z_\xi)$ is invertible and (A2) holds.*

*Proof* From the preceding lemma it follows that $T_{z_\xi}Z_\lambda = \text{Ker}\,(\mathcal{I}''_\lambda[z_\xi])$ and hence $P\mathcal{I}''_\lambda(z_\xi)$ is injective. It remains to prove that $P\mathcal{I}''_\lambda(z_\xi)$ is also surjective. We take $\xi = 0$ and let $W \subset H^1(\mathbb{R})$ be such that $\mathbb{E} = T_z Z_\lambda \oplus W$. For $\mathbf{h} = (h_1, h_2) \in W \times W$ we search $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in W \times W$ such that $\mathcal{I}''_\lambda(z)[\mathbf{u}, \mathbf{v}] = (\mathbf{h}|\mathbf{v})$ for all $\mathbf{v} \in W \times W$. Therefore, $\mathbf{u}$ satisfies the system

$$\begin{aligned} -u_1'' + u_1 - 3U^2_{1-\lambda}u_1 - \lambda u_2 &= \widetilde{h}_1, \\ -u_2'' + u_2 - 3U^2_{1-\lambda}u_2 - \lambda u_1 &= \widetilde{h}_2, \end{aligned} \tag{13.31}$$

where $\widetilde{h}_i = -h_i'' + h_i$, $i = 1, 2$. Setting $\psi = u_1 - u_2$, and $\phi = u_1 + u_2$, we find the decoupled linear system

$$\begin{aligned} -\psi'' + (1+\lambda)\psi - 3U^2_{1-\lambda}\psi &= \widetilde{h}_1 - \widetilde{h}_2, \\ -\phi'' + (1-\lambda)\phi - 3U^2_{1-\lambda}\phi &= \widetilde{h}_1 + \widetilde{h}_2. \end{aligned} \tag{13.32}$$

The first equation can be written as

$$(A_1'(\psi)|v_1) = (\widetilde{h}_1 - \widetilde{h}_2|v_1), \quad \forall\, v_1 \in W, \tag{13.33}$$

where

$$A_1(\psi) = \tfrac{1}{2}\int \left[(\psi')^2 + (1+\lambda)\psi^2 - 3U^2_{1-\lambda}\psi^2\right].$$

Since 0 is not an eigenvalue of $A_1'$, the Fredholm alternative yields a unique $\psi$ solving (13.33).

Similarly, setting

$$A_2(\phi) = \tfrac{1}{2} \int \left[ (\pi')^2 + (1-\lambda)\psi^2 - 3U_{1-\lambda}^2 \phi^2 \right],$$

the second equation of (13.32) becomes $(A_2'(\phi)|v_2) = (h_1 + h_2|v_2)$, for every $v_2 \in W$. Since $\int [(\phi')^2 + (1-\lambda)\phi^2]$ is a norm equivalent to $\|\cdot\|$ and Ker $[A_2']$ is spanned by $U_{1-\lambda}'$, then $h_1 + h_2 \in W = \text{Ker}\,[A_2']^{\perp}$, and the Fredholm alternative yields that $A_2'(\phi) = \widetilde{h}_1 + \widetilde{h}_2$ has a unique solution in $W$. This shows that $P\mathcal{I}_\lambda''(z_\xi)$ is invertible for all $\xi \in \mathbb{R}$.

From the preceding arguments it readily follows that there exists $\delta > 0$ such that $\|PI_\lambda''(z_\xi)\| \geq \delta$ for all $\xi \in \mathbb{R}$.

Setting $\widetilde{\mathbf{w}} = (\widetilde{w}_1, \widetilde{w}_2)$ and using that

$$\mathcal{G}''(z_\xi)[(\mathbf{w}, \widetilde{\mathbf{w}})] = 3\,\varepsilon \int (b_1(x)w^2(x) + b_2(x)\widetilde{w}^2(x))U_{1-\lambda}^2(x+\xi),$$

we get

$$\|\mathcal{G}''(z_\xi)\| \leq \varepsilon\, C, \quad \forall\, \xi \in \mathbb{R},$$

and (A2) follows. □

In order to apply Theorem 5.6.5 it remains to show that the reduced functional $\widetilde{\mathcal{I}}_{\varepsilon,\lambda}(z_\xi) = \mathcal{I}(z_\xi + w_{\varepsilon,s}) + \varepsilon\mathcal{G}(z_\xi + w_{\varepsilon,\xi})$ has a stationary point. According to Remark 5.6.6 we can look for the stationary points of $\mathcal{G}(\xi) := \mathcal{G}(z_\xi)$. One has

$$\mathcal{G}(\xi) = \frac{1}{4} \int (b_1(x)+b_2(x))U_{1-\lambda}^4(x+\xi)dx = \frac{1}{4} \int (b_1(x-\xi)+b_2(x-\xi))U_{1-\lambda}^4(x)dx.$$

Taking into account that $b_1, b_2$ satisfy (13.22) and using arguments already carried out before, one readily verifies that

$$\lim_{|\xi|\to\infty} \mathcal{G}(\xi) = 0.$$

Thus $\mathcal{G}(\xi)$ has at least one maximum or minimum.

The preceding arguments allow us to apply the perturbation result.

**Theorem 13.4.3** *Suppose that* (13.28) *holds and let* $\lambda \in \Gamma$. *Then for* $\varepsilon > 0$ *small enough, the system* (13.27) *has a solution* $(u_\lambda, v_\lambda)$ *near the symmetric state* $(U_{1-\lambda}, U_{1-\lambda})$. □

*Remark 13.4.4* If $\lambda = 0$ the system (13.27) becomes a single NLS equation and we recover the result proved in [13].

*Remark 13.4.5* Linearly coupled systems of nonautonomous NLS equations are discussed in [10].

## 13.5 Appendix

(1) Consider the linear eigenvalue problem (13.30), i.e.

$$A(\psi) := -\psi'' + Q(x)\psi = \nu\psi, \quad \psi \in H^1(\mathbb{R}),$$

where $Q(x)$ is a bounded function such that

$$\lim_{|x|\to\infty} Q(x) = b.$$

It is well known, see e.g. [84, Sect. 3], that the essential spectrum $\sigma_e(A)$ of $A$ is given by

$$\sigma_e(A) = [b, +\infty). \tag{13.34}$$

Moreover, set

$$\Lambda = \inf\left\{\int [(u')^2 + Q(x)u^2] : \int u^2 = 1\right\},$$

and suppose that $\Lambda < b$. Then $\Lambda$ is the smallest eigenvalue of $A$. The corresponding eigenspace is spanned by a positive function and $\Lambda$ is the unique eigenvalue with this property.

(2) Here we carry out in detail the calculation to evaluate (13.25) and prove (13.26). Since $U_{1-\lambda}^2 = 2(1-\lambda) f_\lambda$, one finds that

$$d_\lambda U_{1-\lambda}^2 = 6\lambda f_\lambda - 6\sqrt{1-\lambda}\, x \tanh(\sqrt{1-\lambda}\, x) f_\lambda.$$

Therefore,

$$d_{\lambda,\mathbf{w}}\mathbb{F}(\lambda^*, 0)[u, v] = (h^*, h^*),$$

where

$$\begin{aligned} h^* &= (6\lambda^* - 1) f^* - 6\sqrt{1-\lambda^*}\, x \tanh(\sqrt{1-\lambda^*}\, x) f^* \\ &= \tfrac{13}{5} f^* - 6\sqrt{2/5}\, x \tanh(\sqrt{2/5}\, x) f^*. \end{aligned}$$

One has that $h^* \in Y$ and

$$\begin{aligned} \int h^* f^* &= \tfrac{13}{5} \int \cosh^{-4}(\sqrt{\tfrac{2}{5}}\, x)\, dx \\ &\quad - 6\sqrt{\tfrac{2}{5}} \int x \sinh(\sqrt{\tfrac{2}{5}}\, x) \cosh^{-5}(\sqrt{\tfrac{2}{5}}\, x)\, dx \\ &= \frac{13}{5\sqrt{\frac{2}{5}}} \int \cosh^{-4}(y) dy - \frac{6}{\sqrt{\frac{2}{5}}} \int y \sinh(y) \cosh^{-5}(y) dy. \end{aligned}$$

Integrating by parts, one has

$$\int y \sinh(y) \cosh^{-5}(y) dy = 4 \int \cosh^{-4}(y) dy.$$

Hence

$$\begin{aligned}\int h^* f^* &= \frac{13}{5\sqrt{\frac{2}{5}}} \int \cosh^{-4}(y) dy - \frac{24}{\sqrt{\frac{2}{5}}} \int \cosh^{-4}(y) dy \\ &= -\frac{107}{\sqrt{10}} \int \cosh^{-4}(y) dy < 0,\end{aligned}$$

proving (13.26).

# Appendix A
# Sobolev Spaces

We devote this appendix to the definition and study of the main properties of the Sobolev spaces. The reader can found the details of the proofs in [1, 36, 58, 61, 88]. We assume throughout the appendix that $\Omega$ is an open set in $\mathbb{R}^N$, $k$ is a positive integer and $1 \le p \le +\infty$.

## A.1 Weak Derivative

In this section we introduce the notion of the weak derivative and its main properties. As a motivation of the weak differentiation, we suggest applying integration by parts (or the divergence theorem) to deduce the following characterization.

**Proposition A.1.1** *If $u \in C^1(\Omega)$ and $v_i \in C(\Omega)$, for $i = 1, 2, \ldots, N$, then the following assertions are equivalent:*

1. $v_i = \frac{\partial u}{\partial x_i}$.
2. $\displaystyle\int \varphi(x) v_i(x) dx = -\int u(x) \frac{\partial \varphi}{\partial x_i}(x) dx, \ \ \forall \varphi \in C_0^\infty(\Omega).$ □

**Definition A.1.2** If $u \in L^1_{\text{loc}}(\Omega)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multi-index, we say that $u$ is weakly $\alpha$-derivable if there exists a function $v_\alpha \in L^1_{\text{loc}}(\Omega)$ such that

$$\int \varphi(x) v_\alpha(x) dx = (-1)^{|\alpha|} \int u(x) D^\alpha \varphi(x) dx, \ \ \forall \varphi \in C_0^\infty(\Omega).$$

Clearly, if such a function $v_\alpha$ exists, then it is unique (up to subsets of $\Omega$ with zero measure). This unique function $v_\alpha$ is called the weak $\alpha$-derivative of $u$, and it will be denoted by $v_\alpha = D^\alpha u$.

Taking into account that $C^k(\Omega)$ is the subspace of $C(\Omega)$ which contains all continuous functions $u \in C(\Omega)$ whose (classical) $\alpha$-derivatives with order $|\alpha| \le k$ also belong to $C(\Omega)$, we can construct a similar space $W^k(\Omega)$ for the weak derivative. Specifically, we denote by $W^k(\Omega)$ the set of all functions $u \in L^1_{\text{loc}}(\Omega)$ which are weakly $\alpha$-derivable for every multi-index $\alpha$ with order $|\alpha| \le k$. Clearly, by Proposition A.1.1, $C^k(\Omega) \subset W^k(\Omega)$.

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82, DOI 10.1007/978-0-8176-8114-2, 

Since the weak derivative is linear, i.e.,

$$D^\alpha(u+v) = D^\alpha u + D^\alpha v, \quad D^\alpha(\lambda u) = \lambda D^\alpha u$$

for every weakly $\alpha$-derivable functions $u$ and $v$, and $\lambda \in \mathbb{R}$, $W^k(\Omega)$ is a linear subspace of $L^1_{\text{loc}}(\Omega)$.

We recommend that the reader practice the above definition by verifying the next examples.

*Example A.1.3* 1. If $\Omega = (-1,1) \subset \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ is the modulus function in $\Omega$, i.e., $u(x) = |x|$, for every $x \in \Omega$, then $u \in W^1(\Omega) - C^1(\Omega)$.

2. Let $\Omega = (-1,1) \subset \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ be the sign function in $\Omega$, i.e.,

$$u(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x = 0 \\ -1, & -1 < x < 0, \end{cases}$$

then $u \notin W^1(\Omega)$.

3. Take $\Omega = B(0,R)$, the ball centered at zero with radius $R$, and $u : \Omega \to \mathbb{R}$ a function satisfying $u(x) = f(\,|x|\,), \forall x \in \Omega \setminus \{0\}$ with $f \in C^1((0,R))$. If we have

$$\lim_{r \to 0^+} r^{N-1} f(r) = 0,$$

then

$$u \in W^1(\Omega) \Longleftrightarrow D^\alpha u \in L^1_{\text{loc}}(\Omega), \quad \forall \mid \alpha \mid \leq 1.$$

Using this result, we can prove that the following functions:

$$u(x) = \begin{cases} \dfrac{1}{|x|^\alpha}, & \text{if } 0 < |x| < R, \\ 0, & \text{if } x = 0; \end{cases} \tag{A.1}$$

$$v(x) = \begin{cases} \log\left(\log\left(\frac{4R}{|x|}\right)\right), & \text{if } 0 < |x| < R, \\ 0, & \text{if } x = 0; \end{cases} \tag{A.2}$$

$$w(x) = \begin{cases} |x| \log\left(\log\left(\frac{4R}{|x|}\right)\right), & \text{if } 0 < |x| < R, \\ 0, & \text{if } x = 0; \end{cases} \tag{A.3}$$

verify

$$u \in W^k(\Omega) \Longleftrightarrow k + \alpha < N$$

and $v \in W^{N-1}(\Omega)$, $w \in W^N(\Omega)$.

4. Consider $\Omega = (0,1) \times (-1,1)$, $\Omega_1 = (0,1) \times (0,1)$, $\Omega_2 = (0,1) \times (-1,0)$ and the function $u : \Omega \to \mathbb{R}$ defined by

$$u(x,y) = \begin{cases} 0, & y \le 0, \\ 1, & y > 0. \end{cases}$$

Then $u \notin W^1(\Omega)$, but $u|_{\Omega_1 \cup \Omega_2} \in W^1(\Omega_1 \cup \Omega_2)$.

Note also that if $\alpha$ and $\beta$ are multi-indices and $u \in L^1_{\text{loc}}(\Omega)$ is a weakly $\beta$-derivable function with $\beta$-derivative $D^\beta u$ which is $\alpha$-derivable in turn, then $u$ is weakly $(\alpha+\beta)$-derivable with $D^{\alpha+\beta} u = D^\alpha(D^\beta u)$).[1] This implies that one of the main properties of the spaces $W^k(\Omega)$ is their inductive character; i.e., we can obtain $W^k(\Omega)$ ($k \in \mathbb{N}\setminus\{1\}$) as the functions of $W^{k-1}(\Omega)$ whose derivatives of order $m-1$ belong to $W^1(\Omega)$.

Another remarkable property is that the weak derivative is a local concept.

**Proposition A.1.4** *If $u \in L^1_{\text{loc}}(\Omega)$, then the following assertions are equivalent:*

*(i)* $u \in W^k(\Omega)$.
*(ii) For every $x \in \Omega$ there exists an open neighborhood $V = V(x)$ of $x$ in $\Omega$ such that $u|_V \in W^k(V)$.*

The functions of $W^k(\Omega)$ can be approximated by functions $C_0^\infty(\mathbb{R}^N)$. To do this, we consider $0 \le \rho \in C_0^\infty(\mathbb{R}^N)$ such that its support verifies

$$\text{supp}\, \rho \subset \overline{B(0,1)}$$

and

$$\int_{\mathbb{R}^N} \rho(x)dx = 1.$$

We define for $h > 0$ and $u \in L^1_{\text{loc}}(\Omega)$ the set $\Omega(h) = \{x \in \Omega : h < \text{dist}\,(x, \partial\Omega)\}$ and the function $u_h \in C^\infty(\Omega(h))$ by setting

$$u_h(x) = \frac{1}{h^N} \int \rho\left(\frac{x-y}{h}\right) u(y)dy,$$

for every $x \in \Omega(h)$. For $u \in L^1_{\text{loc}}(\Omega)$, it holds that

$$\{u_h\} \overset{(h\to 0^+)}{\longrightarrow} u \text{ in } L^1_{\text{loc}}(\Omega).$$

If, in addition, $u$ is weakly $\alpha$-derivable (with $\alpha$ a multi-index), then we have $D^\alpha u \in L^1_{\text{loc}}(\Omega)$ and we can consider $(D^\alpha u)_h \in C^\infty(\Omega(h))$. On the other hand, since $u_h \in C^\infty(\Omega(h))$ we can compute its derivative $D^\alpha(u)_h$ in $\Omega(h)$. The theorem of derivation under the integral sign shows that both functions are the same, i.e.,

$$D^\alpha u_h(x) = (D^\alpha u)_h(x), \quad \forall x \in \Omega(h).$$

As a consequence we deduce the following characterization of the weak derivative.

---

[1] In general, $D^{\alpha+\beta}u$ may exists without $D^\beta u$ existing. Indeed, consider the function in Example 4 with $\alpha = (1,0)$ and $\beta = (0,1)$.

**Theorem A.1.5** *If $u, v \in L^1_{\text{loc}}(\Omega)$ and $\alpha$ is a multi-index, then the following assertions are equivalent:*

*(i)* $v = D^\alpha u$.
*(ii) There exists a sequence* $\{\phi_n\} \subset C_0^\infty(\mathbb{R}^N)$ *such that*

$$\{\phi_n\} \longrightarrow u \text{ in } L^1_{\text{loc}}(\Omega),$$
$$\{D^\alpha \phi_n\} \longrightarrow v \text{ in } L^1_{\text{loc}}(\Omega).$$

It is worthwhile to observe that if $u \in L^p(\Omega)$ for $1 \le p < +\infty$, then it is possible to define $u_h$ in all $\mathbb{R}^N$ as being $u_h \in C^\infty(\mathbb{R}^N)$ and

$$\{u_h\} \overset{(h \to 0^+)}{\longrightarrow} u \text{ in } L^p(\Omega).$$

Using this in conjunction with the previous theorem, we obtain the following.

**Corollary A.1.6** *If $u \in L^p(\Omega)$, $v \in L^p_{\text{loc}}(\Omega)$ with $1 \le p < +\infty$ and $\alpha$ a multi-index, then the following assertions are equivalent:*

*(i)* $v = D^\alpha u$.
*(ii) There exists a sequence* $\{\phi_n\} \subset C_0^\infty(\mathbb{R}^N)$ *such that*

$$\{\phi_n\} \longrightarrow u \text{ in } L^p(\Omega),$$
$$\{D^\alpha \phi_n\} \longrightarrow v \text{ in } L^p_{\text{loc}}(\Omega).$$

Theorem A.1.5 also gives a necessary and sufficient condition for the classical derivability of a weakly derivable function. (Remember that every (classically) derivable function is also weakly derivable, i.e., $C^1(\Omega) \subset W^1(\Omega)$).

**Corollary A.1.7** *If $u \in W^1(\Omega)$ satisfies*

$$\frac{\partial u}{\partial x_i} \in C(\Omega) \quad \forall i \in \{1, 2, \ldots, N\},$$

*then $u \in C^1(\Omega)$.*

Of course, by $\frac{\partial u}{\partial x_i} \in C(\Omega)$ we mean that there exists a continuous function in the equivalence class of $\frac{\partial u}{\partial x_i} \in L^1_{\text{loc}}(\Omega)$. Similarly, the condition $u \in C^1(\Omega)$ means also that in the equivalence class of $u \in L^1_{\text{loc}}(\Omega)$ there exists a function of class $C^1$ in $\Omega$.

In turn, a consequence of the previous corollary is the following one.

**Corollary A.1.8** *If $\Omega \subset \mathbb{R}^N$ is open and connected and $u \in W^1(\Omega)$ satisfies $\nabla u = 0$ (a.e. $x \in \Omega$), then $u$ is constant.*

The weak derivative can also be characterized using absolutely continuous functions. Observe that for the local character of the weak derivative it suffices to suppose that $\Omega \subset \mathbb{R}^N$ is a rectangle $\Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_N, b_N)$. If $i \in \{1, 2, \ldots, N\}$, we take $\widehat{\Omega}_i := (a_1, b_1) \times (a_2, b_2) \times \ldots (a_{i-1}, b_{i-1}) \times (a_{i+1}, b_{i+1}) \times \cdots \times (a_N, b_N)$.

We denote by $AC_i(\Omega)$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that the set $B$ of all points $\widehat{x} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \widehat{\Omega}_i$ for which the function $t \mapsto u(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N)$ is absolutely continuous in $(a_i, b_i)$ has zero $(N-1)$-dimensional Lebesgue measure.

**Theorem A.1.9** *If $\Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_N, b_N)$ and $u \in L^1_{\text{loc}}(\Omega)$, then the following assertions are equivalent:*

*(i)* $u \in W^1(\Omega)$.

*(ii) For every $i \in \{1, 2, \ldots, N\}$ there exist a constant $\alpha_i \in \mathbb{R}$, a function $h_i \in L^1_{\text{loc}}(\Omega)$ and a subset $A_i \subset \widehat{\Omega}_i$ such that*

*(a) The $(N-1)$-dimensional Lebesgue measure of $A_i$ is zero.*

*(b) For every $\widehat{x} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \widehat{\Omega}_i - A_i$ we have*

$$u(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N) = \int_{\alpha_i}^{x_i} h_i(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_N) ds + k_{\widehat{x}}$$

*almost everywhere in $x_i \in (a_i, b_i)$ and where $k_{\widehat{x}} \in \mathbb{R}$ denotes a constant which depends on $\widehat{x} \in \widehat{\Omega}_i - A_i$.*

*(iii) For every $i \in \{1, 2, \ldots, N\}$ there exists a function $\widetilde{u}_i \in AC_i(\Omega)$ such that*

*(a) $u(x) = \widetilde{u}_i(x)$ a.e. $x \in \Omega$,*

*(b) The classical derivative[2] $\dfrac{\partial \widetilde{u}_i}{\partial x_i}$ satisfies*

$$\frac{\partial \widetilde{u}_i}{\partial x_i} \in L^1_{\text{loc}}(\Omega).$$

*In addition, if one of the above assertions holds, then*

$$\frac{\partial u}{\partial x_i}(x) = \frac{\partial \widetilde{u}_i}{\partial x_i}(x) = h_i(x) \text{ a.e. } x \in \Omega.$$

*Roughly* the implication (i)⇒(iii) is that *a function $u \in W^1(\Omega)$ if and only if for every coordinate axis, e.g. $x_i$, it is possible to find a function $\widetilde{u}_i$ in the equivalence class of $u \in L^1_{\text{loc}}(\Omega)$ such that $\widetilde{u}_i$ is absolutely continuous for almost everywhere all line segments in $\Omega$ parallel to the coordinate axis and whose partial derivative with respect to $x_i$ is locally integrable in* $\Omega$. Functions satisfying this property were already studied by Beppo Levi and, subsequently, by Leonida Tonelli.

Some consequences of the above characterization are the following.

**Corollary A.1.10** *If $u : \Omega \to \mathbb{R}$ is a locally Lipschitzian function in $\Omega$, then $u \in W^1(\Omega)$.*

Example 3 above shows that the converse is not true.

[2] which exists almost everywhere in $\Omega$.

Next we study whether the product of two weakly derivable functions is weakly derivable. Clearly, the answer is no in general. Indeed, if we consider $\Omega = B(0,1) \subset \mathbb{R}^N$, $\alpha, \beta \in \mathbb{R}$ satisfying

$$\max\{1+\alpha,\ 1+\beta\} < N \ \leq\ 1+\alpha+\beta$$

and $u(x) = \frac{1}{|x|^\alpha}$, $v(x) = \frac{1}{|x|^\beta}$, then $u, v \in W^1(\Omega)$, but $uv \notin W^1(\Omega)$.

However, taking into account that the product of absolutely continuous functions is absolutely continuous, it is possible to prove the following result.

**Corollary A.1.11** *If $u, v \in W^1(\Omega)$, then the following assertions are equivalent:*

*(i)* $u \cdot v \in W^1(\Omega)$ *with* $\frac{\partial}{\partial x_i}(uv) = u\frac{\partial v}{\partial x_i} + v\frac{\partial u}{\partial x_i}$ *for every* $i \in \{1, 2, \ldots, N\}$.
*(ii)* $u \cdot v \in L^1_{\text{loc}}(\Omega)$ *and for every* $i \in \{1, 2, \ldots, N\}$,

$$u\frac{\partial v}{\partial x_i} + v\frac{\partial u}{\partial x_i} \in L^1_{\text{loc}}(\Omega).$$

A particular case in which (ii) (and thus (i)) holds is $u, v \in W^1(\Omega) \cap L^\infty(\Omega)$.

In the sequel we also study the chain rule for the weak derivative. As before for the product, we first see an example proving that it is not true in general. Indeed, if $\Omega = B(0,1) \subset \mathbb{R}^N$, $u(x) = \frac{1}{|x|^\alpha}$ with $1+\alpha < N \leq 1+2\alpha$ and $f(t) = t^2$, $\forall t \in \mathbb{R}$, then $f \in C^1(\mathbb{R}) \subset W^1(\mathbb{R})$ and $u \in W^1(\Omega)$ but $f \circ u \notin W^1(\Omega)$.

Since the composition of a Lipschitz function with an absolutely continuous function is absolutely continuous, the following consequence can be proved.

**Corollary A.1.12** *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $u \in W^1(\Omega)$. Consider $A = \{t \in \mathbb{R} : \exists f'(t)\}$[3] and define the function $g : \mathbb{R} \to \mathbb{R}$ by*

$$g(t) = \begin{cases} f'(t), & t \in A, \\ \\ 0, & t \in \mathbb{R} \setminus A. \end{cases} \tag{A.4}$$

*Then, the composition $f \circ u \in W^1(\Omega)$ with*

$$\frac{\partial}{\partial x_i}(f \circ u)(x) = g(u(x))\frac{\partial u}{\partial x_i}(x),\ a.e.\ x \in \Omega$$

*for every $i \in \{1, 2, \ldots, N\}$.*

We remark that all the hypotheses of the above result are satisfied provided that $f \in C^1(\mathbb{R})$ with bounded derivative.

As an application we obtain the weak derivative of the functions $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$ and $|u|$ provided that $u \in W^1(\Omega)$.

[3] The fundamental theorem of the calculus implies that the measure of $\mathbb{R} \setminus A$ is zero.

**Corollary A.1.13** *If $u \in W^1(\Omega)$, then $u^+, u^-, |u| \in W^1(\Omega)$ with*

$$\nabla u^+(x) = \begin{cases} \nabla u(x), & \text{if } u(x) > 0, \\ 0, & \text{if } u(x) < 0, \end{cases} \quad \text{a.e. } x \in \Omega,$$

$$\nabla u^-(x) = \begin{cases} 0, & \text{if } u(x) > 0, \\ \nabla u(x), & \text{if } u(x) < 0, \end{cases} \quad \text{a.e. } x \in \Omega,$$

$$\nabla |u|(x) = \begin{cases} \nabla u(x), & \text{if } u(x) > 0, \\ 0, & \text{if } u(x) = 0, \\ -\nabla u(x), & \text{if } u(x) < 0, \end{cases} \quad \text{a.e. } x \in \Omega.$$

*In particular,*

$$\nabla u(x) = 0 \text{ a.e. } x \in \Omega_a := \{x \in \Omega \, : \, u(x) = a\},$$

*for every $a \in \mathbb{R}$.*

The second part of the previous corollary is usually attributed to G. Stampacchia (83) (see also [72, Theorem 3.2.2, p. 69]).

## A.2 Sobolev Spaces

In addition to considering an open set $\Omega$ in $\mathbb{R}^N$ and $k \in \mathbb{N}$, we take $p \in [1, +\infty]$.

**Definition A.2.1** The Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) = \left\{ u \in W^k(\Omega) \, : \, D^\alpha u \in L^p(\Omega), \ \ \forall \, |\alpha| \le k \right\}.$$

Clearly, $W^{k,p}(\Omega)$ is a linear subspace of $\left(L^p(\Omega), \|\cdot\|_p\right)$. We can consider two equivalent norms in $W^{k,p}(\Omega)$:

$$\|u\|_{k,p} \equiv \begin{cases} \left[\sum_{|\alpha| \le k} \|D^\alpha u\|_p^p\right]^{1/p} & \text{if } p \in [1, +\infty), \\ \max_{|\alpha| \le k} \|D^\alpha u\|_\infty & \text{if } p = +\infty, \end{cases}$$

and

$$|||u|||_{k,p} \equiv \sum_{|\alpha| \le k} \|D^\alpha u\|_p.$$

In particular, if $p = 2$ the norm $\|\cdot\|_{k,2}$ of the space $H^k(\Omega) \equiv W^{k,2}(\Omega)$ is the one associated to the inner product

$$(u, v)_{k,2} = \sum_{|\alpha| \le k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}.$$

We have to note that from a historical point of view, the introduction of these spaces was not motivated by the similarity with $C^k(\overline{\Omega})$ that we have used in Sect. 1.1.

J. Schauder had already studied the Cauchy problem associated to quasilinear equations of hyperbolic type by applying the theory of fixed points. In order to do this, he considered the space $E$ of the functions $u$ of class $C^k$ in an open $\Omega \subset \mathbb{R}^N$ such that all of its partial derivatives $D^\alpha u$ up to order $k$ belong to $L^2(\Omega)$, and he equipped this space with the norm $\|\cdot\|_{k,2}$. Unfortunately, this space $E$ is not complete with this norm. Just after, Sobolev considered the functions $u \in L^2(\Omega)$ with weak derivatives $D^\alpha u$ in $L^2(\Omega)$ for $|\alpha| \leq k$. This space with the norm $\|\cdot\|_{k,2}$ is already complete.

**Theorem A.2.2** *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space for $1 \leq p \leq +\infty$. In addition,*

*(i)* $W^{k,p}(\Omega)$ *is reflexive for* $1 < p < +\infty$,
*(ii)* $W^{k,p}(\Omega)$ *is separable for* $1 \leq p < +\infty$.

*In particular, $H^k(\Omega)$ is a separable Hilbert space.*

Now it is easy to deduce from Corollaries A.1.11 and A.1.12 the following versions for the product and chain rule in the Sobolev space $W^{k,p}(\Omega)$.

**Proposition A.2.3** *Let $p, q, r \in [1, +\infty]$ be such that*[4]

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

*If $u \in W^{k,p}(\Omega)$ and $v \in W^{k,q}(\Omega)$, then $uv \in W^{k,r}(\Omega)$ with*

$$\frac{\partial(uv)}{\partial x_i}(x) = v(x)\frac{\partial u}{\partial x_i}(x) + u(x)\frac{\partial v}{\partial x_i}(x) \ \ a.\,e.\ x \in \Omega,$$

*for every $i \in \{1, 2, \dots, N\}$.*

**Proposition A.2.4** *Assume that $u \in W^{1,p}(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. Consider the set $A = \left\{t \in \mathbb{R} : \exists f'(t)\right\}$ and the function $g : \mathbb{R} \to \mathbb{R}$ given by* (A.4). *If $f \circ u \in L^p(\Omega)$, then $f \circ u \in W^{1,p}(\Omega)$ with*

$$\frac{\partial(f \circ u)}{\partial x_i}(x) = g(u(x))\frac{\partial u}{\partial x_i}(x)\ a.\,e.\ x \in \Omega$$

*for every $i \in \{1, 2, \dots, N\}$.* □

A sufficient condition to obtain that $f \circ u \in L^p(\Omega)$ is that either $f(0) = 0$ or $\Omega$ is bounded.

It is clear that every function in $C^k(\Omega)$ such that its partial derivatives up to order $k$ are also in $L^p(\Omega)$ belongs to $W^{k,p}(\Omega)$. The next result shows that this class of functions is dense in $W^{k,p}(\Omega)$ provided that $1 \leq p < +\infty$.

**Theorem A.2.5** (N.G. Meyers–J. Serrin) *If $1 \leq p < +\infty$, then the subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.* □

[4] We adopt the agreement $\frac{1}{\infty} = 0$.

The above result is not true for $p = +\infty$. Indeed, the modulus function $u(x) = |x|$ in $(-1, 1)$ belongs to $W^{1,\infty}(-1,1)$ with

$$u'(x) = \begin{cases} 1, & x \geq 0; \\ -1, & x < 0. \end{cases}$$

Since the derivative cannot be approximated in the norm $\|\cdot\|_\infty$ by continuous functions, we deduce that $u$ does not belong to the closure of the subspace $W^{1,\infty}(-1,1) \cap C^1(-1,1)$ in $W^{1,\infty}(-1,1)$.

## A.3 Boundary Values in Sobolev Spaces

**Definition A.3.1** We denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the space $W^{k,p}(\Omega)$. In the particular case $p = 2$, we also write $W_0^{k,p}(\Omega) = H_0^k(\Omega)$.

Clearly, if we consider in $W_0^{k,p}(\Omega)$ the induced norm of $W^{k,p}(\Omega)$, we deduce by Theorem A.2.2 the following result.

**Theorem A.3.2** $W_0^{k,p}(\Omega)$ *is a Banach space provided that* $1 \leq p \leq +\infty$. *In addition,*

*(i)* $W_0^{k,p}(\Omega)$ *is reflexive provided that* $1 < p < +\infty$.
*(ii)* $W_0^{k,p}(\Omega)$ *is separable provided that* $1 \leq p < +\infty$.

*In particular,* $H_0^k(\Omega)$ *is a separable Hilbert space.*

*Remark A.3.3* The reader can verify the following assertions.

1. If $1 \leq p \leq +\infty$ and $u \in W^{k,p}(\Omega)$ has compact support in $\Omega$, then $u \in W_0^{k,p}(\Omega)$.
2. If $1 \leq p < +\infty$ and, for $u \in W_0^{1,p}(\Omega)$, we consider its zero extension $\widetilde{u}$, i.e.,

$$\widetilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$ with

$$\frac{\partial \widetilde{u}}{\partial x_i}(x) = \begin{cases} \dfrac{\partial u}{\partial x_i}(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

for every $i \in \{1, 2, \dots, N\}$.

If we analyze carefully the properties of Sobolev space $W_0^{k,p}(\Omega)$ we see that all of them are similar to the ones satisfied by the space of the functions in $C^1(\overline{\Omega})$ which vanish on the boundary $\partial\Omega$. The next result strengthens this idea.

**Theorem A.3.4** *Consider a function $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.*

*(i) If $u(x) = 0$ for every $x \in \partial\Omega$ then $u \in W_0^{1,p}(\Omega)$.*
*(ii) If $\partial\Omega$ is piecewise of class $C^1$ and $u \in W_0^{1,p}(\Omega)$ then $u(x) = 0$ for every $x \in \partial\Omega$.*

We also have the following version of the chain rule.

**Theorem A.3.5** *If $p \in [1, +\infty)$ and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz with $f(0) = 0$, then $f \circ u \in W_0^{1,p}(\Omega)$ for every $u \in W_0^{1,p}(\Omega)$.* □

Using the space $W_0^{k,p}(\Omega)$, we can define an ordering between values on the boundary of functions in $W^{k,p}(\Omega)$.

**Definition A.3.6** If $1 \leq p < +\infty$ and $u, v \in W^{1,p}(\Omega)$, we say

(i) $u \leq k$ on $\partial\Omega \iff (u-k)^+ = \max\{u-k, 0\} \in W_0^{1,p}(\Omega)$.
(ii) $u \geq k$ on $\partial\Omega \iff -u \leq -k$ on $\partial\Omega$.
(iii) $u \leq v$ on $\partial\Omega \iff u - v \leq 0$ on $\partial\Omega$.
(iv) $u \geq v$ on $\partial\Omega \iff v \leq u$ on $\partial\Omega$.
(v) $u = v$ on $\partial\Omega \iff \begin{cases} u \leq v \text{ on } \partial\Omega \\ \\ v \leq u \text{ on } \partial\Omega. \end{cases}$

*Remark A.3.7* 1. If $\Omega$ has infinite measure, then every nonzero constant does not belong to $W^{1,p}(\Omega)$ and hence the definition given in case (iii) does not cover to the one given in (i).
2. The relation defined in (iii) is an order relation in $W^{1,p}(\Omega)$ (if we understand by *equality on* $\partial\Omega$ that given in (v)).

By using the chain rule a characterization of $W_0^{1,p}(\Omega)$ as the functions in $W^{1,p}(\Omega)$ which vanish on $\partial\Omega$ can be deduced.

**Proposition A.3.8** *If $1 \leq p < +\infty$ and $u \in W^{1,p}(\Omega)$, then*

$$u \in W_0^{1,p}(\Omega) \iff u = 0 \text{ on } \partial\Omega.$$

Applying Theorem A.3.4, we obtain the next connection between weak inequality on $\partial\Omega$ and the classical one.

**Proposition A.3.9** *If $\Omega \subset \mathbb{R}^N$ is open, $p \in [1, +\infty)$ and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, we have:*

*(i) If $u(x) \leq 0$, $\forall x \in \partial\Omega$, then $u \leq 0$ on $\partial\Omega$.*
*(ii) Conversely, if $\partial\Omega$ is piecewise of class $C^1$ and $u \leq 0$ on $\partial\Omega$, then $u(x) \leq 0$ for every $x \in \partial\Omega$.*

The case (ii) of the above proposition is false if the boundary $\partial\Omega$ of $\Omega$ is not smooth. Indeed, it suffices to have in mind that $W_0^{1,p}(B(0,1) \setminus \{0\}) = W_0^{1,p}(B(0,1))$, provided that $N \geq 2$ and $1 \leq p < N$.

The proposition also fails if we replace the constant 0 by a nonzero constant. Indeed, taking $N \geq 5$, $\Omega = \left\{x \in \mathbb{R}^N : |x| > 1\right\}$ and

$$u(x) = \frac{-1}{|\ x\ |^{N-2}}, \quad \forall x \in \overline{\Omega},$$

we have $u \in C^2(\overline{\Omega}) \cap H^1(\Omega)$ and

$$u(x) = -1, \quad \forall x \in \partial\Omega,$$

but it is not true that $u \leq -1$ on $\partial\Omega$.

**Proposition A.3.10** *If* $p \in [1, +\infty)$*, then* $W_0^{k,p}(\mathbb{R}^N) = W^{k,p}(\mathbb{R}^N)$.

*Remark A.3.11* 1. The above result is false for the case $p = \infty$. For example, if $\Omega$ is unbounded in $\mathbb{R}^N$, then every nonzero constant belongs to $W^{k,\infty}(\Omega)$ and it is not in $W_0^{k,\infty}(\Omega)$. Thus, $W_0^{k,\infty}(\Omega) \subsetneq W^{k,\infty}(\Omega)$ in this case.
2. As a consequence of the proposition, we have $u \leq v$ on $\partial\mathbb{R}^N$, for every $u, v \in W^{1,p}(\mathbb{R}^N)$.

To conclude the summary of properties of $W_0^{k,p}(\Omega)$ we give the Poincaré inequality.

**Proposition A.3.12** *If* $p \in [1, +\infty)$ *and* $\Omega \subset \mathbb{R}^N$ *is open and bounded in one direction, then there exists a positive constant C depending uniquely on* $\Omega$ *such that*

$$C\|u\|_p \leq \|\nabla u\|_p, \quad \forall u \in W_0^{1,p}(\Omega).$$

One of the main consequences of the Poincaré inequality is that, under its hypotheses, $\|\nabla u\|_p$ defines a norm $W_0^{1,p}(\Omega)$ which is equivalent to $\|\cdot\|_{1,p}$. In addition, in the case $p = 2$, this new norm $\|\nabla u\|_2$ in $H_0^1(\Omega)$ is associated to the inner product $\int \nabla u \cdot \nabla v$ for $u, v \in H^1(\Omega)$.

## A.4 Embedding Theorems

We now study the well-known Sobolev and Rellich–Kondrachov embedding theorems. Some words are in order to precisely state the meaning of embedding and compact embedding.

**Definition A.4.1** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces.

(i) We say that the space $X$ is embedded in the space $Y$, and we denote it by $X \hookrightarrow Y$, if there exists an injective linear and continuous operator from $X$ into $Y$. In this case, the operator is called an embedding.
(ii) We say that the space $X$ is compactly embedded in the space $Y$, and we denote it by $X \overset{c}{\hookrightarrow} Y$, if there exists an embedding of $X$ in $Y$ which is compact.

Here, we shall consider embeddings of $W^{k,p}(\Omega)$ into three classes of spaces:

(i) $W^{j,q}(\Omega)$ with $0 \leq j \leq k$ ($W^{0,q}(\Omega) \equiv L^q(\Omega)$) and $q$ denotes the conjugate exponent of $p$, i.e., $\left(\frac{1}{q} + \frac{1}{p} = 1\right)$.

(ii) $C_B^j(\Omega)$, for $j \in \mathbb{N} \cup \{0\}$, i.e., the space of functions with continuous and bounded partial derivatives up to order $j$. This is a Banach space with the following norm:

$$\|u\|_{C_B^j(\Omega)} = \max_{0 \leq |\alpha| \leq j} \sup_{x \in \Omega} |D^\alpha u(x)|$$

for every $u \in C_B^j(\Omega)$.

(iii) $C_{B,u}^{j,\nu}(\Omega)$, i.e., the space of functions with bounded and uniformly continuous partial derivatives up to order $j$ in $\Omega$ and such that the partial derivatives of order $j$ satisfy a Hölder condition with exponent $\nu \in (0,1)$. It is also a Banach space with the norm

$$\|u\|_{C_{B,u}^{j,\nu}(\Omega)} = \|u\|_{C_B^j(\Omega)} + \sum_{|\alpha|=j} \sup_{\substack{x,\, y \in \Omega \\ x \neq y}} \frac{|D^\alpha u(x) - D^\alpha(y)|}{|x-y|^\nu},$$

for every $u \in C_{B,u}^{j,\nu}(\Omega)$. Clearly, $C_{B,u}^{j,\nu}(\Omega) \subset C_B^j(\Omega)$.

The embedding of $W^{k,p}(\Omega)$ into a space of the type (i), that is, in $W^{j,q}(\Omega)$, is given by the inclusion $I$ of $W^{k,p}(\Omega)$ in $W^{j,q}(\Omega)$. Indeed, if the inclusion $W^{k,p}(\Omega) \subset W^{j,q}(\Omega)$ holds, then, by the closed graph theorem, the map $I$ is continuous and hence an embedding.

Taking into account that the elements of $W^{k,p}(\Omega)$ are not functions defined in $\Omega$, but are equivalence classes of functions which coincide up to a subset of $\Omega$ with zero measure, we have to make precise the meaning of the embeddings of type (ii) and (iii). It is that the equivalence class $u \in W^{k,p}(\Omega)$ contains a function in the space of the continuous functions which will be the image $Iu$ of the embedding. Thus, for instance, the embedding $W^{k,p}(\Omega) \hookrightarrow C_{B,u}^j(\Omega)$ means that every $u \in W^{k,p}(\Omega)$, considered as a function instead of an equivalence class, can be redefined in a subset in $\Omega$ with zero measure in such a way that the modified function $\widetilde{u}$ (which is equal to $u$ in $W^{k,p}(\Omega)$) belongs to $C_{B,u}^j(\Omega)$ and, for some constant $k > 0$ independent of $u \in W^{k,p}(\Omega)$, satisfies the inequality

$$\|\widetilde{u}\|_{C_{B,u}^j(\Omega)} \leq k\|u\|_{k,p}.$$

The embedding theorems need some hypotheses on the regularity of the boundary $\partial\Omega$ of $\Omega$. If $x \in \mathbb{R}^N$ is a point and $B \subset \mathbb{R}^N$ is an open ball such that $x \notin B$, we call the cone of vertex $x$ and height $r > 0$ to the set

$$C_x = B(x,r) \cap \{x + \lambda(y-x) \,:\, y \in B, \lambda > 0\}.$$

**Definition A.4.2** We say that $\Omega$ satisfies the cone condition if there exists a cone $C$ such that every $x \in \Omega$ is the vertex of a cone $C_x$ contained in $\Omega$ and congruent (by a rigid motion) to $C$.

Every bounded domain $\Omega \subset \mathbb{R}^N$ with boundary $\partial\Omega$ of class $C^1$ satisfies the cone condition. In general the converse assertion is not true (it suffices to consider a square in $\mathbb{R}^2$).

An example of an open set $\Omega \subset \mathbb{R}^2$ which does not satisfy the cone condition is the set of points of the unit ball in $\mathbb{R}^2$ with distance to $(1/2, 0)$ greater than or equal to $1/2$, i.e.,

$$\Omega = B_{\mathbb{R}^2}((0,0),1) - \overline{B}_{\mathbb{R}^2}\left(\left(\frac{1}{2},0\right),\frac{1}{2}\right).$$

We gather in a unique theorem the main embedding results (without taking into account their compactness) of the Sobolev spaces. Usually, it is attributed to S. L. Sobolev, but it also includes improvements due to C.B. Morrey, E. Gagliardo and L. Nirenberg.

**Theorem A.4.3** *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset satisfying the cone condition. Consider also $k \in \mathbb{N} \cup \{0\}$, $1 \le p < \infty$ and $j \in \mathbb{N} \cup \{0\}$. We have the following embeddings.*

1. *If $k < \frac{N}{p}$, then $W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$ for every $q \in \left[p, \frac{Np}{N-kp}\right]$.*
2. *If $k = \frac{N}{p}$, then $W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$ for every $p \le q < \infty$. In addition, in the particular case $p = 1$ and $k = N$, we also have $W^{j+N,1}(\Omega) \hookrightarrow C^j_B(\Omega)$.*
3. *If $k > \frac{N}{p}$, then $W^{j+k,p}(\Omega) \hookrightarrow C^j_B(\Omega)$.*
   *Furthermore, if $\partial\Omega$ is of class $C^1$, then*
4. *If $k - 1 < \frac{N}{p}$, then $W^{j+k,p}(\Omega) \hookrightarrow C^{j,\nu}_{B,u}(\Omega)$ for every $\nu \in \left(0, k - \frac{N}{p}\right]$.*
5. *If $k - 1 = \frac{N}{p}$, then $W^{j+k,p}(\Omega) \hookrightarrow C^{j,\nu}_{B,u}(\Omega)$ for every[5] $\nu \in (0,1)$.*

The following examples show that this theorem is optimal.

*Example A.4.4* Let $\Omega = B(0,R)$ be the ball of center 0 and radius $R > 0$ and fix $k \in \mathbb{N}$. If $1 \le p, q < \infty$ and $\alpha > 0$ are such that $\frac{N}{q} < \alpha < \frac{N-kp}{p}$, then the function $u : \Omega \to \mathbb{R}$ given by (A.1) satisfies $u \in W^{k,p}(\Omega) \setminus L^q(\Omega)$. Indeed, since

$$(\alpha + k)p < N,$$

Example A.1.3-3 shows that $u \in W^{k,p}(\Omega)$. Moreover, $N < \alpha q$ and hence $u \notin L^q(\Omega)$.

This example shows that *the embedding of $W^{k,p}(\Omega)$ in $L^q(\Omega)$ does not hold for $q > \frac{Np}{N-kp}$*. It also proves that *if $kp < N$ then the embedding of $W^{k,p}(\Omega)$ in $C(\Omega)$ is not true*.

*Example A.4.5* Let $k \in \mathbb{N}$ and $p > 1$ such that $kp = N$. Consider $\Omega = B(0,R)$, the ball of radius $R > 0$ centered at zero, and $u : \Omega \to \mathbb{R}$ given by (A.2). Clearly,

---

[5] If $p = 1$ and $N = k - 1$, then it can be $\nu \in (0,1]$.

$u \notin L^\infty(\Omega)$. On the other hand, by Example A.1.3-3, we also have

$$u \in W^{\left[\frac{N}{p}\right],p}(\Omega) = W^{k,p}(\Omega).$$

This example proves that, for $p > 1$ and $kp = N$, even if the embeddings

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [p, +\infty)$$

hold, in general, $W^{k,N/k}(\Omega)$ *is not embedded in* $L^\infty(\Omega)$. The best space in which to embed this space is the Orlicz space of the measurable functions in $\Omega$ satisfying

$$\int_\Omega \left(e^{|u(x)|^{N/(N-k)}}\right) dx < \infty.$$

*Example A.4.6* Consider $k \in \mathbb{N}$, $1 \leq p < \infty$ and $\nu \in (0,1)$. *If* $(k-1) < \frac{N}{p} < k$, *then the embedding of* $W^{k,p}(\Omega)$ *in* $C^\nu_{B,u}(\Omega)$ *is not true for* $\nu > k - \frac{N}{p}$. Indeed, consider $R > 0$, $\Omega = B(0,R)$ and the function $u : \Omega \to \mathbb{R}$ defined by (A.1) with $\alpha \in \mathbb{R}$ such that $\alpha \in \left(-\nu, \frac{N}{p} - k\right)$. Then $(\alpha + k)p < N$ and thus $u \in W^{k,p}(\Omega)$.

In addition, since $\alpha + \nu > 0$,

$$\frac{|u(x) - u(0)|}{|x - 0|^\nu} = | \, x \, |^{-\alpha-\nu}, \ \ \forall x \neq 0$$

and we see that it is not possible to redefine the function $u$ in a set of zero measure in such a way that the new function belongs to $C^\nu(\Omega)$.

*Example A.4.7* Let $k \in \mathbb{N}$ and $p > 1$ be such that $(k-1)p = N$. We prove that *the embedding of case 5 fails for* $\nu = 1$ by showing a function $u \in W^{k,p}(\Omega)$ which cannot be redefined in a zero measure set to belong to $C^1(\Omega)$. Indeed, for $R > 0$, consider the function $u : \Omega = B(0,R) \to \mathbb{R}$ given by (A.3). It is not difficult (see Example A.1.3-3) to verify that

$$u \in W^{\left[\frac{N}{p}\right]+1,p}(\Omega) = W^{k,p}(\Omega).$$

On the other hand, observing that for $x \neq 0$,

$$\frac{|u(x) - u(0)|}{|x - 0|} = \log\left(\log\frac{4R}{|x|}\right)$$

converges to $+\infty$ as $x$ goes to zero, we see that there is no function in the equivalence class of the functions almost everywhere equal to $u$ which belongs to the space $C^1(\Omega)$ (furthermore, this equivalence class does not contain a locally Lipschitzian function in $\Omega$).

Our last example is devoted to show that Theorem A.4.3 is not true in general provided that $\Omega$ is not smooth.

*Example A.4.8* Consider $\Omega = \left\{(x, y) \in \mathbb{R}^2 : 0 < |x| < 1, 0 < y < 1\right\}$ and the function $u$ defined in it by

$$u(x, y) = \begin{cases} 1, & \text{if } 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{if } -1 < x < 0,\ 0 < y < 1. \end{cases}$$

Observe that $u \in W^{1,p}(\Omega)$ for every $p \geq 1$ and that the associated equivalence class does not contain any uniformly continuous function. This proves that *the embedding of $W^{1,p}(\Omega)$ in $C^j_{B,u}(\Omega)$ in case 4 fails for this open $\Omega$ even if we assume that* $0 < \frac{N}{p} < 1$.

F. Rellich had already proved the compactness of some of the previous embeddings in the case $p = 2$. The general case ($p \geq 1$) was studied by V.I. Kondrachov.

**Theorem A.4.9** (Compact embedding of Rellich–Kondrachov) *Let $\Omega$ be a bounded and open subset in $\mathbb{R}^N$ satisfying the cone condition, $k \in \mathbb{N}$ and $p \in [1, +\infty)$. The following embeddings are compact for every $j \in \mathbb{N} \cup \{0\}$.*

1. *If $k < \frac{N}{p}$, then $W^{j+k,p}(\Omega) \overset{c}{\hookrightarrow} W^{j,p}(\Omega)$ for every $q \in \left[1, \frac{Np}{N-kp}\right)$.*
2. *If $k = \frac{N}{p}$, then $W^{j+k,p}(\Omega) \overset{c}{\hookrightarrow} W^{j,p}(\Omega)$ for every $q \in [1, +\infty)$.*
3. *If $k > \frac{N}{p}$, then $W^{j+k,p}(\Omega) \overset{c}{\hookrightarrow} C^j_B(\Omega)$ and $W^{j+k,p}(\Omega) \overset{c}{\hookrightarrow} W^{j,p}(\Omega)$ for every $q \in [1, \infty)$.*

*Furthermore, assuming that $\partial\Omega$ is of class $C^1$, we also have the following assertions.*

4. *If $k > \frac{N}{p}$, then $W^{j+k,p}(\Omega) \overset{c}{\hookrightarrow} C^j_{B,u}(\Omega)$.*
5. *If $k > \frac{N}{p} > k - 1$, then $W^{j+m,p}(\Omega) \overset{c}{\hookrightarrow} C^{j,\nu}_{B,u}(\Omega)$ for every $0 < \nu < k - \frac{N}{p}$.* □

Since the zero extension of a function $u \in W^{k,p}_0(\Omega)$ is an element of $W^{k,p}(\mathbb{R}^N)$, we can consider $W^{k,p}_0(\Omega)$ as a subset of $W^{k,p}(\mathbb{R}^N)$. Therefore, observing that Theorem A.4.3 is satisfied in the case $\Omega = \mathbb{R}^N$, we deduce that it is also true if we replace the spaces $W^{k,p}(\Omega)$ in that theorem by $W^{k,p}_0(\Omega)$ (even if $\Omega$ does not satisfy the cone condition).

Similarly, if $\Omega \subset \mathbb{R}^N$ is a bounded domain, taking an open ball $B(0, R)$ (which satisfies the cone condition) such that $\Omega \subset B(0, R)$, the inclusion $W^{k,p}_0(\Omega) \subset W^{k,p}(B(0, R))$ holds and, by applying Theorem A.4.9 of Rellich–Kondrachov to $W^{k,p}(B(0, R))$, we deduce that the assertions of that theorem are also true if we replace $W^{k,p}(\Omega)$ by $W^{k,p}_0(\Omega)$. Consequently, we have proved the following corollary.

**Corollary A.4.10** *If $\Omega$ is an open subset in $\mathbb{R}^N$, then all embeddings of Theorem A.4.3 hold provided that we replace the space $W^{j+k,p}(\Omega)$ by $W^{j+k,p}_0(\Omega)$.*

*If, in addition $\Omega$ is bounded, then all the compact embeddings in Theorem A.4.9 are also true if we replace the space $W^{j+k,p}(\Omega)$ by $W^{j+k,p}_0(\Omega)$.*

# Exercises

## Exercises related to Chapter 1

1. If $p \in [1, \infty]$ and $k$ is an integer greater than or equal to 1, let $\kappa$ be the number of multi-indices $\alpha$ with order $|\alpha|$ less than or equal to $k$. If $\Omega$ is an open subset of $\mathbb{R}^N$, prove that the set $\{(D^\alpha u)_{|\alpha|\le k} \,:\, u \in W^{k,p}(\Omega)\}$ is a closed subset of the product space $L^p(\Omega) \times \overset{(\kappa)}{\cdots} \times L^p(\Omega)$. As a consequence, show that $W^{k,p}(\Omega)$ is a Banach space which is separable if $1 \le p < +\infty$ and reflexive for $1 < p < +\infty$. (See [36, Proposition 9.1].)
2. Verify the Examples A.1.3.
3. If $X$ and $Y$ are Banach spaces, prove that every linear operator $T : X \longrightarrow Y$ such that $T(A)$ is relatively compact for all bounded set $A \subset X$ is continuous.
4. Prove that the composition of a continuous operator with a compact operator is also a compact operator.
5. Prove that the restriction to $H_0^1(\Omega)$ of the inverse $K$ of the Laplacian operator given in Sect. 1.2.5 is compact from $H_0^1(\Omega)$ into itself.
6. Let $\Omega$ be a bounded subset of $\mathbb{R}^N$. Prove:
   (a) For every two Hölder exponents $0 < \mu < \nu \le 1$, the inclusion $C^{0,\nu}(\overline{\Omega}) \subset C^{0,\mu}(\overline{\Omega})$ is compact.
   (b) Deduce that the restriction to $C^{0,\nu}(\overline{\Omega})$ of the inverse $K$ of the Laplacian operator given in Sect. 1.2.5 is also compact from $C^{0,\nu}(\overline{\Omega})$ into itself.
7. If $X = Y = C(\Omega)$ and $f \in C^1(\mathbb{R})$, show that the Nemitski operator associated to $f$, i.e., $F : X \longrightarrow Y$ given by $F(u) = f \circ u$, $u \in X$, is differentiable and $F'(u)[v] = f'(u)v$, for every $u, v \in X$.
8. Let $a \in L^r(\Omega)$, $b \in L^s(\Omega)$, $r, s \ge 1$, be such that

$$|f(x,u)| \le a(x) + b(x)|u|^{p/q}.$$

   Find $r, s$ in such a way that the Nemitski operator $f$ maps $L^p(\Omega)$ into $L^q(\Omega)$.
9. Let $q_0 > 0$ and $q \in C(\mathbb{R}^N)$ be such that $q(x) > q_0$ for every $x \in \mathbb{R}^N$. Prove that the space

$$\mathcal{E} = \{u \in H^1(\mathbb{R}^N) : \int \left[|\nabla u|^2 + q\, u^2\right] < +\infty\}.$$

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82, DOI 10.1007/978-0-8176-8114-2, 

endowed with the norm

$$\|u\|_{\mathcal{E}}^2 = \int \left[|\nabla u|^2 + q\, u^2\right]$$

is a Banach space such that $\mathcal{E} \subset H^1(\mathbb{R}^N)$ with continuous embedding.

10. If $A \in L(X, Y)$ is invertible, show that the map $F : A \longrightarrow A^{-1}$ is differentiable and $dF(A) : B \longrightarrow -A^{-1}BA^{-1}$. **Hint**. Use the fact that if $T \in L(X, X)$ with $\|T\| < 1$, then $(I - T)$ is invertible and

$$(I - T)^{-1} = \sum (-1)^k T^k.$$

## Exercises related to Chapter 2

11. Let $h \in C([0, 1])$, $k \in C([0, 1] \times [0, 1])$ be continuous functions. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function with constant $L$, prove the following assertions.
    (a) The operator $K : C([0, 1]) \longrightarrow C([0, 1])$ defined by

$$Ku(x) = \int_0^1 k(x, y) f(u(y))\, dy\,, \quad 0 \le x \le 1\,,$$

    is linear and bounded.
    (b) The Hammerstein integral equation

$$u(x) - \int_0^1 k(x, y) f(u(y))\, dy = h(x)\,, \quad 0 \le x \le 1\,,$$

    has a unique solution $u \in C([0, 1])$ provided that $L\, \|K\| < 1$.

12. Let $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function which is increasing in the second variable, i.e., for every fixed $x \in [a, b]$, the function $f(x, y)$ is increasing in $y \in \mathbb{R}$. Prove that if $v, w \in C([a, b])$ satisfy $v(x) \le w(x)$ for every $x \in [a, b]$ and

$$v(x) \le \int_a^b f(x, v(x))\, dx \quad \text{and} \quad \int_a^b f(x, w(x))\, dx \le w(x),$$

for every $x \in [a, b]$, then there exists $u \in C([a, b])$ such that $v(x) \le u(x) \le w(x)$ and

$$u(x) = \int_a^b f(x, u(x))\, dx,$$

for every $x \in [a, b]$.

## Exercises related to Chapter 3

13. Let $k \in C([0,1] \times [0,1])$ be a continuous function and consider the operator $K : C([0,1]) \longrightarrow C([0,1])$ defined by

$$Ku(x) = \int_0^1 k(x,y)u(y)\,dy\,, \quad 0 \le x \le 1\,.$$

Prove that if $\lambda$ belongs to the resolvent $\rho(K)$ of $K$, then there exists $\delta > 0$ such that the integral equation

$$\lambda u(x) = \int_0^1 k(x,y)[u(y) + u(y)^2]\,dy + h(x)\,, \quad 0 \le x \le 1\,,$$

possesses a solution $u \in C([0,1])$ for every $h \in C([0,1])$ satisfying $\|h\|_\infty \le \delta$.

14. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function of class $C^1$ in $\mathbb{R} \setminus \{0\}$ satisfying

$$\varphi(x) = \begin{cases} \frac{1}{n}, & \text{if } \frac{1}{n} - \frac{1}{4n^2} \le x \le \frac{1}{n} + \frac{1}{4n^2}, \\ 0, & \text{if } x = 0, \\ -\frac{1}{n}, & \text{if } -\frac{1}{n} - \frac{1}{4n^2} \le x \le -\frac{1}{n} + \frac{1}{4n^2}\,. \end{cases}$$

Prove that $\varphi$ is differentiable at zero with $\varphi'(0) = 1$, but it is not injective in any neighborhood of zero.

15. Let $\varphi$ be the function defined in Exercise 14 and consider the Nemitski operator associated to it, $F : C([-1,1]) \longrightarrow C([-1,1])$, $F(u) = \varphi \circ u$. Prove that $F$ is differentiable at $u = 0$ with $dF(0)$ equal to the identity, but $F$ is not surjective in any neighborhood of zero.

## Exercises related to Chapter 4

16. Let $X = c_0$ be the Banach space of the real sequences $x = \{x_n\}$ converging to zero with the norm $\|x\| = \max_n |x_n|$. Consider the operator $T : X \longrightarrow X$ defined by

$$(Tx)_1 = \frac{1 + \|x\|}{2} \quad \text{and } (Tx)_{n+1} = x_n\,, \text{ if } n \ge 1.$$

Prove that

(a) $T$ is continuous and maps the unit ball of $X$ into itself,
(b) $T$ is not compact,
(c) $T$ has no fixed point.

Use (c) to deduce that it is not possible to define a topological degree (satisfying all the degree properties) for such a $T$.

17. Let $\Omega$ be a convex bounded set in a Banach space $X$. Suppose that $T : \overline{\Omega} \longrightarrow X$ is compact and $T(\partial\Omega) \subset \Omega$. Prove that $T$ has a fixed point.
18. (Peano's theorem). Let $f$ be a continuous function in a domain $\Omega \subset \mathbb{R}^2$. Prove that for every $(x_0, y_0) \in \Omega$, there exists at least one solution of (2.2). (**Hint**. Use Lemma 2.1.3 and apply the Schauder fixed point Theorem 4.2.6 to the operator $T$ given by (2.5).)
19. (*Improvement of Theorem 4.4.1 under more restrictive conditions*) Let $T : \mathbb{R} \times X \to X$ be a compact map with $X$ a Banach space. Consider the set $\Sigma$ of the pairs $(\lambda, u) \in \mathbb{R} \times X$ which solves the equation $u = T(\lambda, u)$. Assume that $(\lambda_0, u_0) \in \Sigma$ and let $C$ be the connected component $\Sigma$ that contains $(\lambda_0, u_0)$. Prove that if each solution in some neighborhood of $(\lambda_0, u_0)$ is isolated with nonzero index, then all the connected components of $C \setminus \{(\lambda_0, u_0)\}$ are unbounded.
20. Let $E = V \oplus W$ with $V$ finite dimensional. For $R > 0$ we denote by $B_V(R)$ (resp. $S_V(R)$) the closed ball (resp. the sphere) in $V$ of radius $R$ and center 0 and consider

$$\Gamma = \{h : B_V(r) \longrightarrow \mathbb{R} \ : \ h|_{S_V(R)} \text{ is the identity map}\}.$$

By using the Brouwer degree, prove that

$$h(B_V(R)) \cap W \neq \emptyset.$$

## Exercises related to Chapter 5

21. Let $\mathcal{J} : X \longrightarrow \mathbb{R}$ be a convex functional in a Banach space $X$. Prove:
    (a) Every critical point of $\mathcal{J}$ is a global minimum.
    (b) If, in addition, $\mathcal{J}$ is strictly convex, then it has at most a global minimum.
22. (*Weierstrass counterexample*) Prove that the infimum

$$\inf \left\{ \int_{-1}^{1} (tu'(t))^2 \ : \ u \in C^1([-1,1]), \ u(-1) = -1, \ u(1) = 1 \right\}$$

is not attained.
23. Let $X = C[0,1]$,

$$A = \{u \in C[0,1] \ : \ \int_0^{1/2} u - \int_{1/2}^{1} u = 1\}$$

and $\Phi : X \longrightarrow \mathbb{R}$, $\Phi(v) = \|v\|_\infty$. Prove that

$$\inf_{v \in A} \Phi(v) = 1$$

and the above infimum is not attained in $A$. What can be said about the reflexivity of $X$?

24. Let $E$ be a Hilbert space and $M = \mathcal{G}^{-1}(0)$, with $\mathcal{G} \in C^{1,1}(E, \mathbb{R})$ such that $\mathcal{G}'(u) \neq 0$ on $M$. Prove that if $u \in M$ is a local minimum constrained on $M$ for a functional $\mathcal{J} \in C^1(E, \mathbb{R})$, then there exists $\lambda \in \mathbb{R}$ such that $\mathcal{J}'(u) = \lambda \mathcal{G}'(u)$.
25. Let $X$ be a Banach space and $\mathcal{J} \in C^1(X, \mathbb{R})$. Prove that if $\mathcal{J}$ satisfies $(PS)_c$, then $K_c = \{u \in X : \mathcal{J}(u) = c, \ \mathcal{J}'(u) = 0\}$ is a compact set.
26. Prove that if $X$ is a Banach space, then every functional $\mathcal{J} \in C^1(X, \mathbb{R})$ which is bounded from below and satisfies $(PS)_m$ at $m = \inf \mathcal{J}$, attains its infimum. Compare with Corollary 1.2.5.
27. Let $X$ be a Banach space. Prove that every functional $\mathcal{J} \in C^1(X, \mathbb{R})$ bounded from below and satisfying $(PS)_c$ for every $c \in \mathbb{R}$, is coercive. (**Hint**. See [65].)
28. Let $E$ be a Hilbert space and $\mathcal{J} \in C^1(X, \mathbb{R})$ a functional satisfying the Palais–Smale condition $(PS)_c$ at every $c \geq 0$. Assume that $\mathcal{J}(0) = 0$ and that $u = 0$ is a local minimum of $\mathcal{J}$, i.e., $\mathcal{J}(0) \leq \mathcal{J}(u)$ for every $u \in E$ with $\|u\| \leq r$, for some $r > 0$. Use the Ekeland variational principle to prove that if $0 < \rho < r$, then either

$$\inf_{\|u\|=\rho} \mathcal{J}(u) > 0$$

or there is $u_\rho \in E$ such that $\|u_\rho\| = \rho$ and $\mathcal{J}(u_\rho) = 0$. (**Hint**. See [51].)
29. Prove that if the mountain pass critical point is non-degenerate, then its Morse index is 1.

## Exercises related to Chapter 6

30. If $X$ is a Banach space and $L : X \longrightarrow X$ is a linear operator, prove that $\lambda^*$ is a bifurcation point of the equation

$$Lu = \lambda u, \quad u \in X$$

if and only if it is an accumulation point of eigenvalues of $L$.
31. For $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ we consider the equation

$$F(\lambda, u) = 0, \quad u \in \mathbb{R}.$$

   (a) Prove that if $F(\lambda, u) = \lambda u + u^3$, then $\lambda = 0$ is the unique bifurcation point of the equation.
   (b) Prove that there is no bifurcation point if $F(\lambda, u) = \lambda^2 u + u^3$.

32. The following example shows that Theorem 6.1.2 fails if it is only imposed that the geometrical multiplicity of the eigenvalue $\lambda^*$ is one. Consider $E = \mathbb{R}^2$ and $F : \mathbb{R} \times E \longrightarrow \mathbb{R}$ given by

$$F(\lambda, x, y) = \begin{pmatrix} \lambda x - y \\ \lambda y \end{pmatrix}, \quad (x, y) \in E.$$

Prove:

(a) $\lambda^* = 0$ is an eigenvalue of $L(x, y) = \begin{pmatrix} -y \\ 0 \end{pmatrix}$ with algebraic multiplicity two and geometrical multiplicity one.
(b) The equation $F(\lambda, x, y) = 0$, $(x, y) \in E$ has no bifurcation points.

## Exercises related to Chapters 7–13

33. Prove that if $f$ satisfies (7.2), then the functional $\mathcal{J}_\lambda$ considered in Sect. 7.1.2 is of class $C^1$.
34. Give a detailed proof of Proposition 8.1.1.
35. Prove that zero is the unique solution of (11.8) for $\lambda \geq \lambda_1$.
36. Let $f \in C^1(\mathbb{R})$ satisfy $f(0) = 0$, $\lambda_1 < f'(0) < \lambda_2$ and suppose that $\lim_{u\to\pm\infty} \frac{f(u)}{u} := \gamma_\pm < \lambda_1$. Prove that (8.9) has exactly one positive solution and one negative solution.
37. By using Banach contraction Theorem 2.1.2 (see Remark 8.3.5), prove the uniqueness of solution in Theorem 8.3.3 provided that item 2 of Theorem 8.3.2 holds.
38. (*A variational proof of the uniqueness of solution in Theorem 8.3.2*) By applying Exercise 21 to the functional considered in Sect. 8.3.3, prove that if the $C^1$ function $f$ satisfies $f'(u) < \lambda_1$, for every $u \in \mathbb{R}$, then (8.1) has a unique solution in $H_0^1(\Omega)$ for all $h \in L^2(\Omega)$.
39. Prove the claim of Remark 8.4.11.
40. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function with $f(0) = 0$ and $u$ a solution of

$$\begin{aligned} -\Delta u &= f(u), \quad x \in \Omega \\ u &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{13.2}$$

Prove:

(a) $\lambda u$ is a sub-solution for every $\lambda > 1$.
(b) $\lambda u$ is a super-solution if $0 < \lambda < 1$.

41. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $\frac{f(t)}{t}$ is increasing. Prove that every sub-solution $u_1 > 0$ and super-solution $u_2 > 0$ of (13.2) satisfy

$$u_1 \not< u_2, \quad \text{in } \Omega.$$

42. Let $g \in C^2(\overline{\Omega})$ be a function such that

(a) The sets $\Omega^+ := \{x \in \Omega : g(x) > 0\}$ and $\Omega^- := \{x \in \Omega : g(x) < 0\}$ are not empty.
(b) $\Gamma := \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega$, and $\nabla g(x) \neq 0$ on $\Gamma$.

Assume also that $1 < p < 2^* - 1$, $\Omega$ is a bounded regular domain in $\mathbb{R}^N$ with $N \geq 3$, and that $m \in L^\infty(\Omega)$ changes sign in $\Omega^+$. Prove that the closure of the set of nontrivial weak solutions of problem

$$\begin{aligned} -\Delta u &= \lambda m(x)u + g(x)|u|^{p-1}u, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned}$$

contains a bounded continuum of positive solutions bifurcating from the point $(\lambda_1(m), 0)$ and also from $(\lambda_{-1}(m), 0)$. In particular, there is at least one positive solution of the problem for every $\lambda \in (\lambda_{-1}(m), \lambda_1(m))$. (**Hint**. See [40].)

43. (*Semi-positon problem*) Consider the problem (9.1) where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $\lambda > 0$ and $f \in C^1([0, +\infty))$ such that $f(0) < 0$ and $f(s) = m_\infty s + g(s)$ with $\lim_{s\to+\infty} g(s)/s = 0$. Prove that if (9.4) holds, then there exists $\delta > 0$ such that (9.1) has at least one positive solution for $\lambda \in (\frac{\lambda_1}{m_\infty} - \delta, \frac{\lambda_1}{m_\infty})$. Prove similarly the existence of solution to the right of $\frac{\lambda_1}{m_\infty}$ when the inequality (9.4) is reversed.
44. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying

$$\int_{-\infty}^{+\infty} f(s) < \infty \text{ and } \lim_{|u|\to\pm\infty} f(u) = 0.$$

Consider the functional $\mathcal{J}$ defined in $H_0^1(\Omega)$ by

$$\mathcal{J}(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{\lambda_1}{2}\int u^2 - \int F(u),\ u \in H_0^1(\Omega),$$

where $F(u) = \int_0^u f$. Prove:
(a) $\mathcal{J}$ is bounded from below.
(b) By applying Theorem 5.3.8, it has either a global minimum or a mountain pass critical point.

45. Prove Remark 11.1.5.
46. Assume that $1 < p < 2^* - 1$ and consider the *Nehari manifold* $N = \{u \in H_0^1(\Omega) : \int |\nabla u|^2 = \int |u|^{p+1}\}$. Prove that
(a) The infimum $m$ of $\mathcal{J}(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1}$ on $N$ is attained at a nonzero solution of the problem

$$\begin{aligned} -\Delta u &= |u|^{p-1}u, && x \in \Omega \\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

(b) Verify that $m$ is the mountain pass value of $\mathcal{J}$:

$$m = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : h(0) = 0,\ \mathcal{J}(\gamma(1)) \le 0\}$.

47. Use the Pohozaev identity to prove that every eigenfunction $\varphi \not\equiv 0$ of the Laplace operator satisfies

$$\frac{\partial\varphi}{\partial n} \not\equiv 0, \quad \text{in } \partial\Omega.$$

48. Let $\lambda > 0$ and $0 < q < 1 < p < 2^* - 1$, where $2^*$ is given by (7.3). Prove that there exists $\lambda_0 > 0$ such that the problem

$$\begin{aligned} -\Delta u &= \lambda |u|^{q-1}u + |u|^{p-1}u, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned}$$

has no solution for $\lambda > \lambda_0$, at least one solution for $\lambda = \lambda_0$ and at least two positive solutions for $\lambda < \lambda_0$. (**Hint**. See [9].)
49. Prove that, with obvious changes, the results of Chap. 12 are also true for the case that the dimension $N = 2$.
50. Prove Lemma 12.4.1.
51. Assume that the potential $V$ satisfies $(V1)$–$(V2)$ of Sect. 13.2 and $V(x) - 1 \equiv c > 0$ for every $|x| \geq 1$. Verify that the abstract method in Sect. 5.6 can be applied. (**Hint**. Write the auxiliary equation $PI'_\varepsilon(z_\xi + w) = 0$ as

$$w = -(PI''_\varepsilon(z_\xi))^{-1}[PI'_\varepsilon(z_\xi) + R(z_\xi, w)] := N_{\varepsilon,\xi}(w)$$

and apply Theorem 2.1.2.)

# Bibliography

[1] Adams, R.A., *Sobolev Spaces*, Academic Press, Boston, MA, 1975.
[2] Amann, H., *Lectures on some fixed point theorems*, *Monograf. Mat., Inst. Mat. Pura Apl.*, Rio de Janeiro, 1974.
[3] Amann, H., Ambrosetti, A. and Mancini, G., *Elliptic equations with noninvertible Fredholm linear part and bounded nonlinearities*, *Math. Z.* **158** (1978), 179–194.
[4] Amann, H. and Hess, P., *A multiplicity result for a class of elliptic boundary value problems*, *Proceedings of the Royal Society of Edinburgh*, **84A** (1979), 145–151.
[5] Amann, H. and Weiss, S., *On the uniqueness of the topological degree*, *Math. Z.* **130** (1973), 39–54.
[6] Ambrosetti, A., *Branching points for a class of variational operators*, *J. d'Analyse Math.* **76** (1998), 321–335.
[7] Ambrosetti, A., *Remarks on some systems of nonlinear Schroedinger equations*, *J. Fixed Point Theory Appl.* **4** (2008), no. 1, 35–46.
[8] Ambrosetti, A., Badiale, M. and Cingolani, S., *Semiclassical states of nonlinear Schrödinger equations*, *Arch. Rational Mech. Anal.* **140** (1997), 285–300.
[9] Ambrosetti, A., Brezis, H. and Cerami, G., *Combined effects of concave and convex nonlinearities in some elliptic problems*, *J. Funct. Anal.* **122** (1994), 519–543.
[10] Ambrosetti, A., Cerami, G. and Ruiz, D., *Solitons of linearly coupled systems of semilinear non-autonomous equations on* $\mathbb{R}^n$, *J. Funct. Anal.* **254** (2008), 2816–2845.
[11] Ambrosetti, A. and Colorado, E., *Standing waves of some coupled nonlinear Schrödinger equations*, *Jour. London Math. Soc.* **75** (2007), 67–82.
[12] Ambrosetti, A., Colorado, E. and Ruiz, D., *Multi-bump solitons to linearly coupled NLS systems*, *Calc. Var. PDE* **30** (2007), 85–112.
[13] Ambrosetti, A., García Azorero, J. and Peral, I., *Elliptic variational problems in* $R^n$ *with critical growth*, *J. Differential Equations*, **168** (2000), 10–32.
[14] Ambrosetti, A. and Hess, P., *Positive solutions of asymptotically linear elliptic eigenvalue problems*, *J. Math. Anal. Appl.* **73** (2) (1980), 411–422.
[15] Ambrosetti, A. and Malchiodi, A., *Nonlinear Analysis and Semilinear Elliptic Problems*. Cambridge University Press 2007.
[16] Ambrosetti, A. and Prodi, G., *On the inversion of some differentiable mappings with singularities between Banach spaces*, *Ann. Mat. Pura ed Appl.* **93** (1972), 231–246.
[17] Ambrosetti, A. and Prodi, G., *A Primer on Nonlinear Analysis*, Cambridge University Press, London, 1993.
[18] Ambrosetti, A. and Rabinowitz P.H., *Dual variational methods in critical points theory and applications*, *J. Funct. Anal.* **14** (1973), 349–381.
[19] Arcoya, D. and Boccardo L., *Critical points for multiple integrals of calculus of variations*, *Arch. Rat. Mech. Anal.* **134** 3 (1996), 249–274.
[20] Arcoya, D. and Carmona, J., *Quasilinear elliptic problems interacting with its asymptotic spectrum*, *Nonlinear Anal.* **52** (2003) no. 6, 1591–1616.

A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82, DOI 10.1007/978-0-8176-8114-2, 

[21] Arcoya, D. and Carmona, J., *On the so-called Ambrosetti problems*, *J. European Math. Soc.* **8** (2006), 181–188.

[22] Arcoya, D., Carmona, J. and Martinez-Aparicio, P.J., *Bifurcation for quasilinear elliptic singular BVP*, *Comm. PDE*, **36** (2011), 670–692.

[23] Arcoya, D., Carmona, J. and Pellacci, B., *Bifurcation for Quasilinear Operator*, *Proc. Roy. Soc. Edinb.* **131A** (2001), 733–765.

[24] Arcoya, D. and Gámez, J.L., *Bifurcation theory and related problems: Anti-maximum principle and resonance*, *Comm. in PDE*, **26** (9 & 10) (2001), 1879–1911.

[25] Arcoya, D. and Rossi, J., *Antimaximum principle for quasilinear problems*, *Adv. Diff. Equat.* **9**/9-10 (2004), 1185–1200.

[26] Arcoya, D. and Ruiz, D., *The Ambrosetti–Prodi problem for the p-Laplace operator*, *Comm. PDE* **31** (2006), 849–865.

[27] Arcoya, D. and Segura, S., *Uniqueness of solutions for some elliptic equations with a quadratic gradient term*, ESAIM: Control, Optimization and Calculus of Variations (2008).

[28] Arias, M. and Campos, J., *Radial Fučik spectrum of the Laplace operator*, *J. Math. Analysis and Appl.* **190** (1995), 654–666.

[29] Aubin, J.P. and Ekeland, I., *Applied Nonlinear Analysis*, Wiley-Interscience, New York, 1984.

[30] Benci, V. and Rabinowitz, P.H., *Critical point theorems for indefinite functionals*, *Invent. Math.* **52** (1979), 241–273.

[31] Bhöme, R., *Die Lösung der Verzweigungsgleichungen für nichtlineare Eigenwertprobleme*, *Math Z.* **128** (1972), 105–126.

[32] Boccardo, L., Gallouët, T. and Murat, F., *A unified presentation of two existence results for problems with natural growth*, *127-137 Research Notes in Mathematics* **296**, Pitman 1993.

[33] Boccardo, L., Murat, F. and Puel, J.P., *Existence de solutions non bornees pour certaines équations quasi-linéaires*, *Portugaliae Mathematica,* **41** (1982), 507–534.

[34] Boccardo, L., Murat, F. and Puel, J.P., *Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique*, *Nonlinear partial differential equations and their applications*. Collège de France Seminar, Vol. **IV** (Paris, 1981/1982), 19–73, Res. Notes in Math., **84**, Pitman, Boston, 1983.

[35] Boccardo, L., Murat, F. and Puel, J.P., *Résultats d'existence pour certains problèmes elliptiques quasilinéaires*, *Ann. Scuola Norm. Sup. Pisa*, **11** (1984), 213–235.

[36] Brezis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[37] Brezis, H and Nirenberg, L., *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, *Comm. on Pure and Applied Math.* **36** (1983), 437–477.

[38] Brezis, H. and Nirenberg, L., *$H^1$ versus $C^1$ local minimizers*, *Comptes rendus de l'Académie des sciences. Série 1, Mathématique* **317** 5 (1993), 465–472.

[39] Canino, A., *Multiplicity of solutions for quasilinear elliptic equations, Topol. Meth. in Nonlinear Anal.* **6** (1995), 357–370.

[40] Cingolani, S. and Gámez, J.L., *Positive solutions of a semilinear elliptic equation on $\mathbb{R}^N$ with indefinite nonlinearity*, *Adv. Differential Equations*, **1** (1996), 773–791.

[41] Clement, P. and Peletier, L.A., *An anti-maximum principle for second-order elliptic operators*, *J. Diff. Equations* **34** (1979), 218–229.

[42] Corvellec, J.-N., Degiovanni, M. and Marzocchi, M., *Deformation properties for continuous functionals and critical point theory*, *Topol. Methods Nonlinear Anal.* **1** 1 (1993), 151–171.

[43] Dancer, E.N., *On the ranges of certain weakly nonlinear elliptic partial differential equations*, *J. Math. Pures et Appl.* **57** (1978), 351–366.

[44] Deimling, K., *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[45] del Pino, M. and Felmer, P., *Local mountain passes for semilinear elliptic problems in unbounded domains*, *Calc. Var. PDE* **4** (1996), 121–137.

[46] Ekeland, I., *On the variational principle*, *J. Math. Anal. Appl.*, **47** (1974), 324–353.

[47] De Figueiredo, D.G., *Positive solutions of semilinear elliptic problems. Differential equations (Sao Paulo, 1981), Lecture Notes in Math.*, 957, Springer, Berlin-New York, 1982, pp. 34–87.

[48] De Figueiredo, D.G., *The Ekeland Variational Principle with applications and detours*, Springer, New York, 1989.

[49] De Figueiredo, D.G. and Gossez, J.-P., *On the first curve of the Fučik spectrum of an elliptic operator*, *Diff. and Integral Equat.* **7** (1994), 1285–1302.

[50] De Figueiredo, D.G., Lions, P.L. and Nussbaum, R.D., *A priori estimates and existence of positive solutions for semi-linear elliptic equations*, *Journal Math. Pures et Appl.* **61** (1982), 41–63.

[51] De Figueiredo, D.G. and Solimini, S., *A variational approach to superlinear elliptic problems*, *Comm. PDE* **9** (1984), 699–717.

[52] Floer, A. and Weinstein, A., *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, *J. Funct. Anal.* **69** (1986), 397–408.

[53] Fučik, S., *Remarks on a result by A. Ambrosetti and G. Prodi*, *Boll. U.M.I.* **11** (1975), 259–267.

[54] Gallouët, T and Kavian, O., *Resultats d'existence et de non-existence pour certain problemes demi-lineaires a l'infini*, *Ann. Fac. Sci. Toulouse,* **3** (1981), 201–246.

[55] Gámez, J.L. and Ruiz, J.F., *Bifurcation of solutions of elliptic problems: Local and global behaviour*, *Topol. Meth. in Nonl. Anal.* **23** (2004), 203–212.

[56] Gámez, J.L. and Ruiz, J.F., *Sharp estimates for the Ambrosetti-Hess problem and consequences*, *J. Eur. Math. Soc.* **8**(2) (2006), 287–294.

[57] Gidas, B. and Spruck, J., *A priori bounds for positive solutions of nonlinear elliptic equations*, *Comm. PDE* **6** (8) (1981), 883–901.

[58] Gilbarg, D. and Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

[59] Kesavan, S., *Symmetrization & Applications*, *Series in Analysis*, Vol. 3, World Scientific, 2006.

[60] Krasnoselskii, M., *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan, New York, 1964.

[61] Kufner, A., John, O. and Fucik, S., *Function Spaces*, Noordhoff International Publishing, Leyden, 1977.

[62] Landesman, E. and Lazer, A., *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, *J. Math. Mech.* **19** (609) (1970), 609–623.

[63] Leoni, G., *Existence of solutions for holonomic dynamical systems with homogeneous boundary conditions. Nonlinear Anal.* **23** 4, (1994), 427–445.

[64] Leray, J. and Schauder, J., *Topologie et équations fonctionelles*, *Ann. Scient. Éc. Norm. Sup.* **51** (1934), 45–78.

[65] Li, S., *An existence theorem on multiple critical points and its applications*, *Acta Math. Sci.* **4** (1984).

[66] Lyapunov, A.M., *Sur les figures d'équilibre peu différents des ellipsoides d'une masse liquide homogène douée d'un mouvement de rotation*, *Zap. Akad. Nauk St. Petersburg* (1906), 1–225.

[67] Lyapunov, A.M., *Problème général de la stabilité du mouvement*, *Ann. Fac. Sci. Toulouse* **2** (1907), 203–474.

[68] Marino, A. and Prodi, G., *Metodi perturbativi nella teoria di Morse*, *Boll. Un. Mat. Ital.* (4) 11 (1975), 1–32.

[69] Mawhin, J. and Willem, M., *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, Berlin, 1989.

[70] McKenna, P.J. and Walter, W.,*On the multiplicity of the solution set of some nonlinear boundary value problems*, *Nonlinear Analysis, T.M.A.* **8** (1984), 893–907.

[71] Milnor, J.W., *Topology from the Differentiable Viewpoint*, *Princeton University Press*, Princeton, NJ, 1997.

[72] Morrey, C.B. Jr. *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, New York, 1966.

[73] Orsina, L. and Puel, J.-P., *Positive solutions for a class of nonlinear elliptic problems involving quasilinear and semilinear terms*, *Commun. PDE* **26**, 1665–1689 (2001).

[74] Rabinowitz, P.H., *Some global results for nonlinear eigenvalue problems*, *J. Funct. Anal.* **7** (1971), 487–513.

[75] Rabinowitz, P.H., *On bifurcation from infinity*, *J. Diff. Equations* **14** (1973), 462–475.

[76] Rabinowitz, P.H., *The mountain pass theorem: theme and variations*, *Differential equations (Sao Paulo, 1981)*, pp. 237–271, *Lecture Notes in Math.*, 957, Springer, Berlin-New York, 1982.

[77] Rabinowitz, P.H., *Minimax methods in critical point theory with applications to differential equations*, *CBMS Regional Conference Series in Mathematics*, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[78] Rabinowitz, P.H., *On a class of nonlinear Schrödinger equations*, *Z. Angew. Math. Phys.* **43** (1992), 270–291.

[79] Ruiz, D. and Suárez, A. *Existence and uniqueness of positive solution of a logistic equation with nonlinear gradient term*, *Proc. Roy. Soc. Edinburgh Sect. A* **137** (2007), 555–566.

[80] Schauder, J., *Der Fixpunktsatz in Funktionalräumen*, *Studia Math.* **2** (1930), 171–180

[81] Schmidt, E., *Zur Theory der linearen und nichtlinearen Integralgleichungen, 3 Teil*, *Math. Annalen* **65** (1908), 370–399.

[82] Stampacchia, G., *Le Problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinuous*, *Ann. Inst. Fourier Grenoble.* **117** (1965), 138–152.

[83] Stampacchia, G., *Equations elliptiques du second ordre à coefficients discontinuous*, Presses Univ. Montreal, 1966.

[84] Stuart, C.A., *An introduction to elliptic equations on* $R^N$, *Nonlinear functional analysis and applications to differential equations* (Trieste, 1997), pp. 237–285, World Sci. Publ., River Edge, NJ, 1998.

[85] Sweers, G., $L^N$ *is sharp for the antimaximum principle*, *J. Diff. Equations* **134** (1997), 148–153.

[86] Wang, X., *On concentration of positive bound states of nonlinear Schrödinger equations*, *Comm. Math. Phys.*, **153** (1993), 229–244.

[87] Whyburn, G.T., *Topological Analysis*, *Princeton Univ. Press*, Princeton, N.J. 1958.

[88] Ziemer, W.P., *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*, *Springer*, Berlin, 1989.

# Index



A. Ambrosetti, D. Arcoya, *An Introduction to Nonlinear Functional Analysis and Elliptic Problems,* Progress in Nonlinear Differential Equations and Their Applications 82, DOI 10.1007/978-0-8176-8114-2, 